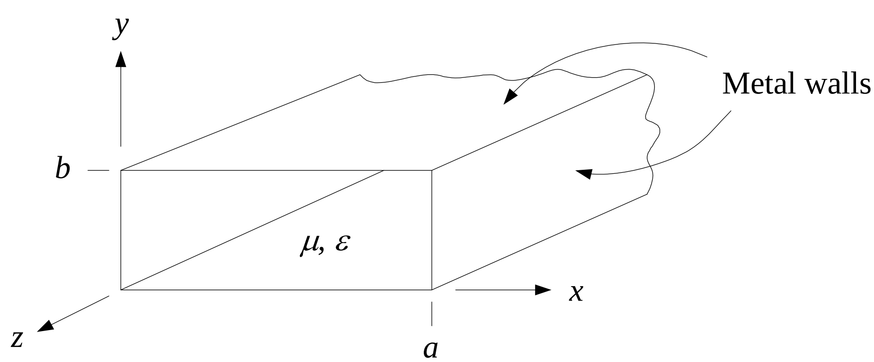
Lecture 10: TEM, TE, and TM Modes for Waveguides. Rectangular Waveguide.

We will now generalize our discussion of transmission lines by considering EM waveguides. These are "pipes" that guide EM waves. Coaxial cables, hollow metal pipes, and fiber optical cables are all examples of waveguides.

We will assume that the waveguide is invariant in the *z*-direction:



and that the wave is propagating in *z* as $e^{-j\beta z}$. (We could also have assumed propagation in –*z*.)

Types of EM Waves

We will first develop an extremely interesting property of EM waves that propagate in homogeneous waveguides. This will lead to the concept of "modes" and their classification as

- Transverse Electric and Magnetic (TEM),

- Transverse Electric (TE), or
- Transverse Magnetic (TM).

Proceeding from the Maxwell curl equations:

$$\nabla \times \bar{E} = -j\omega\mu\bar{H} \quad \Rightarrow \quad \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -j\omega\mu\bar{H}$$

or

$$\hat{x}: \quad \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x$$

$$\hat{y}: \quad -\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) = -j\omega\mu H_y$$

$$\hat{z}: \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z$$

However, the spatial variation in z is known so that

$$\frac{\partial\left(e^{-j\beta z}\right)}{\partial z} = -j\beta\left(e^{-j\beta z}\right)$$

Consequently, these curl equations simplify to

$$\frac{\partial E_z}{\partial y} + j\beta E_y = -j\omega\mu H_x \qquad \text{(3.3a),(1)}$$

$$-\frac{\partial E_z}{\partial x} - j\beta E_x = -j\omega\mu H_y \qquad \text{(3.3b),(2)}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \qquad \text{(3.3c),(3)}$$

We can perform a similar expansion of Ampère's equation $\nabla \times \bar{H} = j\omega\varepsilon\bar{E}$ to obtain

$$\frac{\partial H_z}{\partial y} + j\beta H_y = j\omega\varepsilon E_x \qquad (3.4a),(4)$$

$$-j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \qquad (3.4b),(5)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z \qquad (3.5c),(6)$$

Now, (1)-(6) can be manipulated to produce simple algebraic equations for the transverse (x and y) components of $\bar{E}$ and $\bar{H}$. For example, from (1):

$$H_x = \frac{j}{\omega\mu}\left(\frac{\partial E_z}{\partial y} + j\beta E_y\right)$$

Substituting for E_y from (5) we find

$$H_x = \frac{j}{\omega\mu}\left[\frac{\partial E_z}{\partial y} + j\beta\frac{1}{j\omega\varepsilon}\left(-j\beta H_x - \frac{\partial H_z}{\partial x}\right)\right]$$

$$= \frac{j}{\omega\mu}\frac{\partial E_z}{\partial y} + \frac{\beta^2}{\omega^2\mu\varepsilon}H_x - \frac{j\beta}{\omega^2\mu\varepsilon}\frac{\partial H_z}{\partial x}$$

or,

$$H_x = \frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right) \qquad (3.5a),(7)$$

where $k_c^2 \equiv k^2 - \beta^2$ and $k^2 = \omega^2\mu\varepsilon$. (3.6)

Similarly, we can show that

$$H_y = -\frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta\frac{\partial H_z}{\partial y}\right) \qquad (3.5b),(8)$$

$$E_x = \frac{-j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right) \qquad (3.5c),(9)$$

$$E_y = \frac{j}{k_c^2}\left(-\beta\frac{\partial E_z}{\partial y} + \omega\mu\frac{\partial H_z}{\partial x}\right) \qquad (3.5d),(10)$$

Most important point: From (7)-(10), we can see that **all transverse** components of $\bar{E}$ and $\bar{H}$ can be determined from **only the axial** components E_z and H_z. It is this fact that allows the mode designations TEM, TE, and TM.

Furthermore, we can use superposition to reduce the complexity of the solution by considering each of these mode types separately, then adding the fields together at the end.

TE Modes and Rectangular Waveguides

A transverse electric (TE) wave has $E_z = 0$ and $H_z \neq 0$. Consequently, all $\bar{E}$ components are transverse to the direction of propagation. Hence, in (7)-(10) with $E_z = 0$, then all transverse components of $\bar{E}$ and $\bar{H}$ are known once we find a solution for only H_z. Neat!

For a rectangular waveguide, the solutions for E_x, E_y, H_x, H_y, and H_z are obtained in Section 3.3 of the text. The solution and the solution process are interesting, but not needed in this course.

What is found in that section is that

$$k_{c,mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \qquad \begin{matrix} m,n = 0,1,\ldots \\ (m = n \neq 0) \end{matrix} \tag{11}$$

Therefore, from (3.6)

$$\beta = \beta_{mn} = \sqrt{k^2 - k_{c,mn}^2} \tag{12}$$

These m and n indices indicate that only discrete solutions for the transverse wavenumber (k_c) are allowed. Physically, this occurs because we've bounded the system in the x and y directions. (A vaguely similar situation occurs in atoms, leading to shell orbitals.)

Notice something important. From (11), we find that $m = n = 0$ means that $k_{c,00} = 0$. In (7)-(10), this implies infinite field amplitudes, which is not a physical result. Consequently, the $m = n = 0$ TE (or TM) modes are not allowed.

One exception *might* occur if $E_z = H_z = 0$ (a transverse electric and magnetic, i.e., TEM, wave) since this leads to indeterminate forms in (7)-(10). However, it can be shown that inside hollow metallic waveguides when both $m = n = 0$ and $E_z = H_z = 0$, then $\bar{E} = \bar{H} = 0$. This means there is no TEM mode in these hollow metallic waveguides.

Consequently, EM waves will propagate in hollow metallic waveguides only when the frequency is "large enough" since the TEM mode cannot exist.

To understand this concept, consider (12) and using $k^2 = \omega^2 \mu\varepsilon$

$$\beta = \sqrt{\omega^2 \mu\varepsilon - k_{c,mn}^2} \underbrace{=}_{(11)} \sqrt{\omega^2 \mu\varepsilon - \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right]} \qquad (13)$$

If $m = n = 0$ is not allowed in the hollow metallic waveguide (i.e., no TEM mode), then ω must be "large enough" so that β is a real number as required for a propagating mode. Otherwise β will be imaginary ($\beta \to -j\alpha$), leading to pure attenuation and no propagation of the wave $e^{-j\beta z} \to e^{-\alpha z}$.

This turns out to be a general result. That is, for a hollow conductor waveguide, EM waves will propagate only when the frequency is large enough and exceeds some lower threshold. This minimum frequency for wave propagation is called the cutoff frequency $f_{c,mn}$.

It can be shown that guided EM waves require at least two distinct conductors in order to support wave propagation all the way down to 0^+ Hz.

The cutoff frequencies for TE modes in a rectangular waveguide are determined from (13) with $\beta = 0$ to be

$$f_{c,mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \qquad \begin{matrix} m,n = 0,1,\ldots \\ (m = n \neq 0) \end{matrix} \qquad (3.84),(14)$$

In other words, these are the frequencies where $\beta_{mn} = 0$ and wave propagation begins when the frequency slightly exceeds $f_{c,mn}$.

For an X-band rectangular waveguide, the cross-sectional dimensions are *a* = 2.286 cm and *b* = 1.016 cm. Using (14):

TE$_{m,n}$ Mode Cutoff Frequencies

m	n	$f_{c,mn}$ (GHz)
1	0	6.562
2	0	13.123
0	1	14.764
1	1	16.156

In the X-band region (≈ 8.2-12.5 GHz), only the TE_{10} mode can propagate in the waveguide regardless of how it is excited. (We'll also see shortly that no TM modes will propagate either.) This is called single mode operation and is most often the preferred application for hollow waveguides.

On the other hand, at 15.5 GHz any combination of the first three of these modes ***could*** exist and propagate inside a metal, rectangular waveguide. Which combination actually exists will depend on how the waveguide is excited.

Note that the TE_{11} mode (and all higher-ordered TE modes) could not propagate. (We'll also see next that no TM modes will propagate at 15.5 GHz either.)

TM Modes and Rectangular Waveguides

Conversely to TE modes, transverse magnetic (TM) modes have $H_z = 0$ and $E_z \neq 0$.

The expression for the cutoff frequencies of TM modes in a rectangular waveguide

$$f_{c,mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \qquad m,n = 1,2\ldots \qquad (15)$$

is very similar to that for TE modes given in (14).

It can be shown that if **either** $m = 0$ or $n = 0$ for TM modes, then $\bar{E} = \bar{H} = 0$. This means that no TM modes with $m = 0$ or $n = 0$ are allowable in a rectangular waveguide.

For an X-band waveguide from (15):

$TM_{m,n}$ Mode Cutoff Frequencies

m	n	$f_{c,mn}$ (GHz)
1	1	16.156
1	2	30.248
2	1	19.753

Therefore, no TM modes can propagate in an X-band rectangular waveguide when $f < 16.156$ GHz.

Dominant Mode

Note that from 6.56 GHz $\leq f \leq$ 13.12 GHz in the X-band rectangular waveguide, only the TE_{10} mode can propagate. This mode is called the dominant mode of the waveguide.

See Fig 3.9 in the text for plots of the electric and magnetic fields associated with this mode.

TEM Mode

The transverse electric and magnetic (TEM) modes are characterized by $E_z = 0$ **and** $H_z = 0$.

In order for this to occur, it can be shown from (3.4) and (3.5) that it is necessary for $f_c = 0$. In other words, there is no cutoff frequency for waveguides that support TEM waves.

Rectangular, circular, elliptical – actually, any shaped hollow, metallic waveguides – **cannot** support TEM waves.

It can be shown that at least two separate conductors are required for TEM waves. Examples of waveguides that will allow TEM modes include coaxial cable, parallel plate waveguide, stripline, and microstrip.

Microstrip is the type of microwave circuit interconnection that we will use in this course. This “waveguide” will support the “quasi-TEM” mode, which like regular TEM modes has no non-zero cutoff frequency.

However, if the frequency is large enough, other modes (i.e., “higher-order” modes) will begin to propagate on a microstrip. This is usually not a desirable situation.