Lecture 7: Sinusoidal Steady State, Phasors.

In some of our studies of time varying electromagnetic fields, we will be considering sinusoidal steady state signals.

In this case, the use of phasors greatly simplifies the analysis since in Maxwell's equations

$$\frac{\partial}{\partial t} \to j\omega$$

assuming an $e^{j\omega t}$ time dependence. In particular, we usually assume a $\cos(\omega t)$ time dependence by default.

What is a Phasor?

To answer this question, imagine we have a sinusoidally varying function of time

$$f(t) = f_0 \cos(\omega t + \phi)$$

By Euler's identity

$$e^{jx} = \cos x + j \sin x$$

then

$$f(t) = \mathrm{Re}\left[f_0 e^{j(\omega t + \phi)} \right]$$

or

$$f(t) = \mathrm{Re}\left[\underbrace{f_0 e^{j\phi}}_{F_0} e^{j\omega t} \right]$$

The quantity F_0 in this last expression is what is called a **phasor**. That is, the phasor F_0 is a very simple and compact method of representing the:

- amplitude, and
- phase angle (i.e., time delay with respect to the source) of the function $f(t)$.

These two properties are all that is needed to represent in a shorthand notation, of sorts, the time variation of a function in a linear, time invariant system that has sinusoidal steady state excitation.

In electromagnetics, our functions are generally vectors as well as phasors. Additionally, these "**vector phasors**" are functions of space. Because of this, analyzing such problems can be complicated.

Example N7.1: Determine the vector phasor representation of

$$\bar{B}(y,t)=\hat{a}_x B_0\cos(\omega t-\beta y)$$

$$\bar{B}(y,t)=\text{Re}\left[\hat{a}_x B_0 e^{j(\omega t-\beta y)}\right]=\text{Re}\left[\underbrace{\hat{a}_x B_0 e^{-j\beta y}}_{\text{vector phasor}} e^{j\omega t}\right]$$

$$\Rightarrow \quad \bar{B}(y)=\hat{a}_x B_0 e^{-j\beta y},$$

which is the vector phasor representation of $\bar{B}(y,t)$.

Example N7.2: Determine the phasor representation for Faraday's law

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

Assuming a $\cos(\omega t)$ time response, then $\bar{E}(t) = \text{Re}\left[\bar{E}e^{j\omega t}\right]$ and $\bar{B}(t) = \text{Re}\left[\bar{B}e^{j\omega t}\right]$. In these two expressions, $\bar{E}$ and $\bar{B}$ in the Re operators are vector phasors.

Substituting these into Faraday's law gives

$$\nabla \times \text{Re}\left[\bar{E}e^{j\omega t}\right] = -\frac{\partial}{\partial t}\text{Re}\left[\bar{B}e^{j\omega t}\right]$$

But, the Re operator commutes with the differentiation operator. Therefore,

$$\text{Re}\left[\left(\nabla \times \bar{E}\right)e^{j\omega t} + \frac{\partial}{\partial t}\left(\bar{B}e^{j\omega t}\right)\right] = 0$$

or

$$\text{Re}\left[\underbrace{\left(\nabla \times \bar{E} + j\omega \bar{B}\right)}_{\text{phasor}}e^{j\omega t}\right] = 0$$

Consequently, the phasor representation of Faraday's law is

$$\nabla \times \bar{E} = -j\omega \bar{B}$$

Maxwell's Equations in Phasor Form

Applying the result of this last example, we can easily write Maxwell's equations in phasor form as

$$\nabla \times \bar{E} = -j\omega\bar{B} \qquad \nabla \cdot \bar{D} = \rho_v$$

$$\nabla \times \bar{H} = j\omega\bar{D} + \bar{J} \qquad \nabla \cdot \bar{B} = 0$$

and the continuity equation

$$\nabla \cdot \bar{J} = -j\omega\rho_v$$

Complex Numbers Aren't Necessarily Phasors

Finally, note that a phasor is generally a complex number, but not every complex number is a phasor! For example, in circuit analysis:

Phasors	**Not phasors**
V	P
I	Q (reactive power)
	Z_C
	Z_L
	H (transfer fct.)
	L
	C
	R

Similarly, in electromagnetics:

Phasors	**Not phasors**
$\bar{E}$	P
$\bar{D}$	$\bar{S}$ (Poynting vector)
$\bar{B}$	ε
$\bar{H}$	μ
ρ_v	η
$\bar{J}$	

Remember that a phasor is a shorthand representation for a function that has the following two time domain properties:

1. sinusoidal time dependence,
2. zero time average value.