

## Lecture 6: Maxwell's Equations, Boundary Conditions.

In the last four lectures, we have been investigating the behavior of dynamic (i.e., time varying) electric and magnetic fields.

In the previous lecture, we discussed “Maxwell’s law” (i.e., Ampère’s law with the added displacement current term). For a capacitor, we found that displacement current completes the path of the current where conduction ends.

Notice in the definition of capacitor displacement current

$$I_d = C \frac{dV}{dt}$$

that a time varying electric field in space is producing a conduction current, which subsequently **produces a time varying magnetic field**. Amazing!

Conversely, in Faraday’s law

$$emf = -\frac{d\psi_m}{dt}$$

a magnetic field effect (more specifically, a time varying magnetic flux) produces an emf (a “source” voltage).

This is a beautiful “**duality**” between these two effects:

- Maxwell's law:  $\frac{\partial \bar{E}(t)}{\partial t} \xrightarrow{\text{induces}} \bar{B}(t)$
- Faraday's law:  $\frac{\partial \bar{B}(t)}{\partial t} \xrightarrow{\text{induces}} \bar{E}(t)$

Since a time varying electric field produces a magnetic force and vice versa, we now speak of an **electro-magnetic** field, rather than electric and magnetic fields separately.

Because of this duality, we will see shortly that **electromagnetic signals can propagate as waves!** It is because of this fantastic circumstance that there exists light, radio communications, satellite remote sensing, RADAR, fiber optic networks, CAT scans, etc.

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## Maxwell's Equations

The laws of classical electromagnetics can be neatly summarized into a concise collection called **Maxwell's equations**.

In **point form**, Maxwell's equations read:

$\nabla \times \bar{E}(t) = -\frac{\partial \bar{B}(t)}{\partial t}$	Faraday's law
$\nabla \times \bar{H}(t) = \frac{\partial \bar{D}(t)}{\partial t} + \bar{J}(t)$	Ampère's law
$\nabla \cdot \bar{D}(t) = \rho_v(t)$	Gauss' law, I
$\nabla \cdot \bar{B}(t) = 0$	Gauss' law, II

In **integral form**, Maxwell's equations read

$\oint_{c(s)} \bar{E}(t) \cdot d\bar{l} = -\frac{d}{dt} \int_{s(c)} \bar{B}(t) \cdot d\bar{s}$	Faraday's law
$\oint_{c(s)} \bar{H}(t) \cdot d\bar{l} = \frac{d}{dt} \int_{s(c)} \bar{D}(t) \cdot d\bar{s} + \int_{s(c)} \bar{J}(t) \cdot d\bar{s}$	Ampère's law
$\oint_{s(v)} \bar{D}(t) \cdot d\bar{s} = \int_{v(s)} \rho_v(t) dv$	Gauss' law, I
$\oint_{s(v)} \bar{B}(t) \cdot d\bar{s} = 0$	Gauss' law, II

In addition, the **continuity equation** (conservation of charge) reads in point form:

$$\nabla \cdot \bar{J}(t) = -\frac{\partial \rho_v(t)}{\partial t}$$

and in integral form

$$\oint_{s(v)} \bar{J}(t) \cdot d\bar{s} = -\frac{d}{dt} \int_{v(s)} \rho_v(t) dv$$

These laws describe all of classical (i.e., non-quantum mechanical) electromagnetism. Maxwell's equations are an amazingly short and concise set of equations. However, these equations are usually difficult to solve for real-world problems.

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## Interdependent Equations

As it turns out, **not all of these equations are independent** for dynamic fields. For example, if we take the divergence of Ampère's law:

$$\nabla \cdot (\nabla \times \bar{H}) = 0 = \nabla \cdot \bar{J} + \nabla \cdot \left( \frac{\partial \bar{D}}{\partial t} \right)$$

we find that it reduces to

$$\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$$

which is the **continuity equation**.

There are other examples of interdependencies among Maxwell's equations for dynamic fields.

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## Constitutive Equations

For dynamic electromagnetism, the static **constitutive equations** are still applicable:

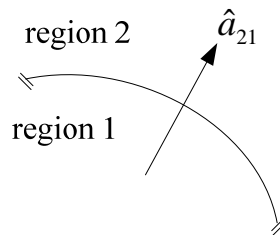
- $\bar{D}(t) = \varepsilon \bar{E}(t)$
- $\bar{B}(t) = \mu \bar{H}(t)$
- $\bar{J}(t) = \sigma \bar{E}(t)$

However, for **sinusoidal steady state** problems, the material parameters are often a function of frequency. That is

$$\varepsilon = \varepsilon(\omega) \quad \mu = \mu(\omega) \quad \sigma = \sigma(\omega)$$

## Boundary Conditions

The **boundary conditions** for dynamic EM fields remain the same as were derived earlier in EE 381 for static fields:



**Tangential components** –

- $\hat{a}_{21} \times [\bar{E}_2(t) - \bar{E}_1(t)] = 0$
- $\hat{a}_{21} \times [\bar{H}_2(t) - \bar{H}_1(t)] = \bar{J}_s (= \bar{K})$

**Normal components** –

- $\hat{a}_{21} \cdot [\bar{D}_2(t) - \bar{D}_1(t)] = \rho_s(t)$
- $\hat{a}_{21} \cdot [\bar{B}_2(t) - \bar{B}_1(t)] = 0$

**Example N6.1:** The electric field  $\bar{E}(x,t) = \hat{a}_z E_o \cos(\omega t + \beta x)$  V/m exists in free space. Determine  $\bar{H}(t)$  and  $\beta$  consistent with this electric field and all of Maxwell's equations.

- From Faraday's law:  $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$ . For this example

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} (=0) & \frac{\partial}{\partial z} (=0) \\ 0 & 0 & E_z \end{vmatrix} = -\hat{a}_y \frac{\partial E_z}{\partial x} = -\frac{\partial \bar{B}}{\partial t}$$

Therefore,

$$\begin{aligned} \frac{\partial B_y}{\partial t} &= \frac{\partial E_z}{\partial x} = \frac{\partial}{\partial x} [E_o \cos(\omega t + \beta x)] \\ &= E_o (-1) \sin(\omega t + \beta x) \cdot \frac{\partial}{\partial x} (\omega t + \beta x) \end{aligned}$$

or 
$$\frac{\partial B_y}{\partial t} = -\beta E_o \sin(\omega t + \beta x)$$

So, 
$$B_y = \frac{\beta}{\omega} E_o \cos(\omega t + \beta x) + C$$

The constant  $C$  cannot be a function of time. It is often taken as zero for dynamical problems if there are no sources present for constant magnetic fields. Therefore, since  $\bar{B} = \mu \bar{H}$  then

$$\bar{H}(t) = \hat{a}_y \frac{\beta}{\omega \mu_0} E_o \cos(\omega t + \beta x) \text{ [A/m]}$$

- From Ampere's law:  $\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} + \bar{J}^0$ . Then

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} (=0) & \frac{\partial}{\partial z} (=0) \\ 0 & H_y & 0 \end{vmatrix} = \hat{a}_z \frac{\partial H_y}{\partial x} = \epsilon_0 \frac{\partial \bar{E}}{\partial t}$$

Therefore,

$$\frac{\partial}{\partial x} \left[ \frac{\beta}{\omega \mu_0} E_o \cos(\omega t + \beta x) \right] = \epsilon_0 \frac{\partial}{\partial t} \left[ \underbrace{E_o \cos(\omega t + \beta x)}_{=E_z} \right]$$

or

$$\frac{\beta}{\omega \mu_0} E_o (-1) \sin(\omega t + \beta x) \underbrace{\frac{\partial(\omega t + \beta x)}{\partial x}}_{=\beta} = \epsilon_0 E_o (-1) \sin(\omega t + \beta x) \underbrace{\frac{\partial(\omega t + \beta x)}{\partial t}}_{=\omega}$$

So that

$$\frac{\beta^2}{\omega \mu_0} = \epsilon_0 \omega \quad \text{or} \quad \beta = \pm \omega \sqrt{\mu_0 \epsilon_0} \quad [\text{rad/m}]$$

- $\nabla \cdot \bar{D} = \nabla \cdot (\epsilon_0 \bar{E}) = \epsilon_0 \frac{\partial E_z}{\partial z} = 0$ . ✓ consistent with Maxwell's equations.
- $\nabla \cdot \bar{B} = \frac{\partial B_y}{\partial y} = 0$ . ✓ consistent with Maxwell's equations.