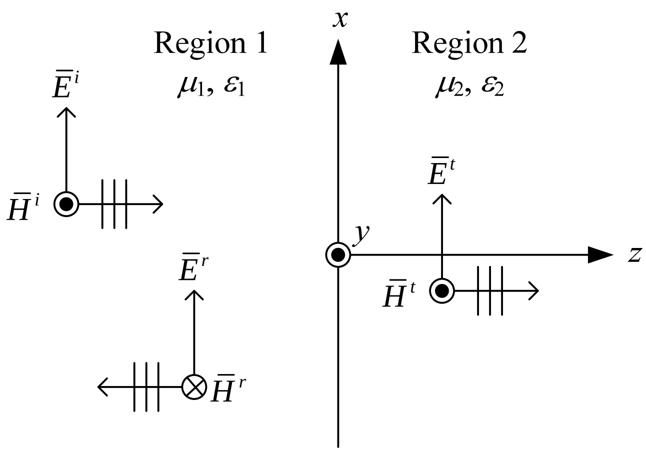
Lecture 30: Example of a Normally Incident UPW on a Lossless Half Space.

Example N30.1. A UPW is incident from free space onto a glass half space with $\varepsilon_r = 4$ and $\mu_r = 1$, as shown in the figure below. It is specified that

$$\bar{E}^i = \hat{a}_x 1 \cdot e^{-j\beta_1 z} \quad \text{V/m}$$

and $f = 200$ MHz.



(a) Determine the time domain $\bar{E}$ and $\bar{H}$ for the incident, reflected, and transmitted fields.

$$\eta_1 = \eta_0 \quad \text{and} \quad \eta_2 = \eta_0 \sqrt{\frac{\mu_r}{\varepsilon_r}} = \eta_0 \sqrt{\frac{1}{4}} = \frac{\eta_0}{2} \ \Omega$$

Therefore $$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\frac{1}{2} - 1}{\frac{1}{2} + 1} = -\frac{1}{3}$$

and $$\mathrm{T} = 1 + \Gamma = \frac{2}{3}$$

Hence, $$\bar{E}^i(z,t) = \hat{a}_x 1 \cdot \cos(\omega t - \beta_1 z)$$

where $\omega = 2\pi f = 2\pi \cdot 2\times 10^8 = 4\pi\times 10^8$ rad/s

and $\beta_1 = \beta_0 = \omega\sqrt{\mu_0\varepsilon_0} = \dfrac{4\pi\times 10^8}{3\times 10^8} = \dfrac{4\pi}{3}$ rad/m

Therefore, for the incident fields:

$$\bar{E}^i(z,t) = \hat{a}_x \cos\left(4\pi\times 10^8 t - \frac{4\pi}{3}z\right) \quad \text{V/m}$$

and $$\bar{H}^i(z,t) = \hat{a}_y \frac{1}{\eta_0}\cos\left(4\pi\times 10^8 t - \frac{4\pi}{3}z\right) \quad \text{A/m}$$

where $\eta_0 = 376.73037\ \Omega$. We now have the complete time domain expressions for the incident $\bar{E}$ and $\bar{H}$.

For the reflected fields:

$$\bar{E}^r(z,t) = \hat{a}_x E_1^+ \Gamma \cos(\omega t + \beta_1 z)$$
$$= -\hat{a}_x \frac{1}{3}\cos\left(4\pi\times 10^8 t + \frac{4\pi}{3}z\right) \quad \text{V/m}$$

and

$$\bar{H}^r(z,t) = -\hat{a}_y \frac{E_1^+\Gamma}{\eta_1}\cos(\omega t + \beta_1 z)$$
$$= \hat{a}_y \frac{1}{3\eta_0}\cos\left(4\pi\times 10^8 t + \frac{4\pi}{3}z\right) \quad \text{A/m}$$

Note that in turns out $\bar{E}^r$ points in the $-\hat{a}_x$ direction, which is opposite our initial assumption in the figure above. But with $\bar{H}^r$ pointing in the $\hat{a}_y$ direction, the cross product $\bar{E}\times\bar{H}$ points in the proper direction of propagation, which is the $-\hat{a}_z$ direction for the reflected wave.

For the transmitted fields in region 2 we have:

$$\bar{E}^t(z,t) = \hat{a}_x E_1^+ \mathrm{T}\cos(\omega t - \beta_2 z)$$

With $\beta_2 = \beta_0\sqrt{\mu_{r2}\varepsilon_{r2}} = 2\beta_0 = \dfrac{8\pi}{3}$ rad/m

then $\bar{E}^t(z,t) = \hat{a}_x \dfrac{2}{3}\cos\left(4\pi\times 10^8 t - \dfrac{8\pi}{3}z\right)$ V/m

For the transmitted magnetic field:

$$\bar{H}^t = \hat{a}_y \frac{E_1^+\mathrm{T}}{\eta_2}\cos(\omega t - \beta_2 z) = \hat{a}_y \frac{1\cdot 2/3}{\eta_0/2}\cos(\omega t - \beta_2 z)$$

$$= \hat{a}_y \frac{4}{3\eta_0}\cos\left(4\pi\times 10^8 t - \frac{8\pi}{3}z\right) \quad \text{A/m}$$

(b) Compute the time average power transmitted through a 5-m^2 surface of the glass.

The time average Poynting vector in region 2 is given as

$$\bar{S}_{av2} = \frac{1}{2}\mathrm{Re}\left(\bar{E}_2 \times \bar{H}_2^*\right)$$

which has only a z component. Since $\bar{E}_2$ and $\bar{H}_2$ are not functions of x or y, the total power transmitted though this surface area is simply

$$P_{av,z} = \text{Area}\cdot S_{av2,z}$$

$$= 5\cdot\frac{1}{2}\frac{\left|E^t\right|^2}{\eta_2} = 5\cdot\frac{1}{2}\frac{(2/3)^2}{\eta_0/2} = 5.899 \quad \text{mW}$$

(c) Compute the **standing wave ratio (SWR)**.

In region 1: $$\text{SWR} = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+\frac{1}{3}}{1-\frac{1}{3}} = 2$$

In region 2: $$\text{SWR} = 1$$

From the *VisualEM* "Example 6.8" Worksheet

- In the first plot of the worksheet ("Total Ex in regions 1 and 2") it is apparent that there is more oscillation in E_x per unit z in region 2 ($z > 0$) than in region 1 ($z < 0$).

  Why? Because the wavelength is *smaller* in region 2 than region 1:
  $$\lambda_1 = \frac{2\pi}{\beta_1} = \frac{2\pi}{4\pi/3} = \frac{3}{2} \text{ m}$$
  $$\lambda_2 = \frac{2\pi}{\beta_2} = \frac{2\pi}{8\pi/3} = \frac{3}{4} \text{ m}$$
  A smaller wavelength indicates a more electromagnetically "dense" material.

- The next plot in the worksheet shows that $\bar{E}_1$ is the sum of incident and reflected waves.

- The final plot is an animation of the total electric field in both regions 1 and 2. Study this animation carefully and notice:

  ✓ The wave "pulsates" in region 1. This is what partial wave interference looks like in the time domain. There is no pulsation in region 2 since there is no interference there.

  ✓ In region 1:

$$E_{\text{max}} = E_1^+ \left(1 + |\Gamma|\right)$$

$$E_{\text{min}} = E_1^+ \left(1 - |\Gamma|\right)$$

$$\frac{E_{\text{max}}}{E_{\text{min}}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \text{SWR} = 2$$

Example 6.8

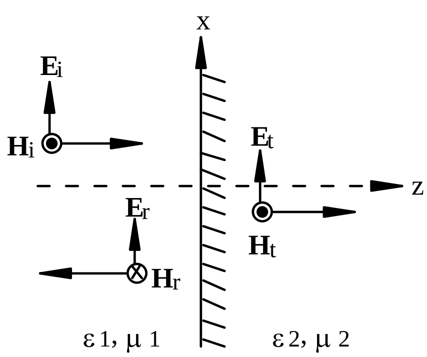
A Uniform Plane Wave Normally Incident on a Lossless Half Space

Purpose

To compute and visualize the incident, reflected and transmitted electric fields that result when a uniform plane wave is normally incident on a lossless half space. The concept of a standing wave is also discussed in this worksheet and the standing wave ratio is computed.

Enter parameters

As given in Example 6.8, a uniform plane wave (UPW) is assumed to be **normally** incident on a half space as shown in the figure below. Both half spaces are assumed to be lossless.



Choose the parameters of this wave and the two half spaces:

$f := 200 \cdot 10^{6}$		Frequency (Hz).
$E_m := 1$		Amplitude on the incident UPW (V/m).
$\varepsilon_{r1} := 1$	$\mu_{r1} := 1$	Relative ε and μ of half space 1.
$\varepsilon_{r2} := 4$	$\mu_{r2} := 1$	Relative ε and μ of half space 2.

Compute the material parameters for both half spaces:

$$\varepsilon_0 := 8.854 \cdot 10^{-12} \qquad \mu_0 := 4 \cdot \pi \cdot 10^{-7}$$

$$\varepsilon_1 := \varepsilon_{r1} \cdot \varepsilon_0 \qquad \mu_1 := \mu_{r1} \cdot \mu_0$$

$$\varepsilon_2 := \varepsilon_{r2} \cdot \varepsilon_0 \qquad \mu_2 := \mu_{r2} \cdot \mu_0$$

Compute the radian frequency (ω), period (T_p), phase constants (β_1, β_2) and intrinsic impedances (η_1, η_2) for the two half spaces:

$$\omega := 2\cdot\pi\cdot f \qquad T_p := \frac{1}{f}$$

$$\beta_1 := \omega\cdot\sqrt{\mu_1\cdot\varepsilon_1} \qquad \beta_2 := \omega\cdot\sqrt{\mu_2\cdot\varepsilon_2}$$

$$\eta_1 := \sqrt{\frac{\mu_1}{\varepsilon_1}} \qquad \eta_2 := \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

Compute the total E field in both regions

We will now determine the total electric field in both regions 1 and 2 in the figure shown above. To do this, we will first compute the reflection and transmission coefficients for the UPW using (115) and (116) in Chap. 6 of the text:

$$\Gamma := \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad \Gamma = -0.3333$$ Reflection coefficient.

$$T := \frac{2\cdot\eta_2}{\eta_2 + \eta_1} \qquad T = 0.6667$$ Transmission coefficient.

Referring to Example 6.8 in the text, the time-domain form of the incident, reflected and transmitted electric fields are then:

$$E_{ix}(z,t) := \text{if}\left(z \le 0, E_m\cdot\cos\left(\omega\cdot t - \beta_1\cdot z\right), 0\right)$$ Incident electric field (V/m).

$$E_{rx}(z,t) := \text{if}\left(z \le 0, \Gamma\cdot E_m\cdot\cos\left(\omega\cdot t + \beta_1\cdot z\right), 0\right)$$ Reflected electric field (V/m).

$$E_{tx}(z,t) := \text{if}\left(z \ge 0, T\cdot E_m\cdot\cos\left(\omega\cdot t - \beta_2\cdot z\right), 0\right)$$ Transmitted electric field (V/m).

The total electric field in region 1 is the sum of the incident and reflected electric field:

$$E_{1x}(z,t) := E_{ix}(z,t) + E_{rx}(z,t)$$

whereas in region 2, the total electric field is simply the transmitted field (there is no reflected fields in this half space):

$$E_{2x}(z,t) := E_{tx}(z,t)$$

The total electric field throughout space is then:

$$E_x(z,t) := \text{if}\left(z \le 0, E_{1x}(z,t), E_{2x}(z,t)\right)$$

Plot E_x versus z

For the remainder of this worksheet, we will generate a number of plots and animation clips to illustrate the behavior of the electric fields for a UPW that is reflected and transmitted from an infinite and lossless half space.

The first plot we will generate is the total E_x field as a function of z but at a fixed time. Choose an instant of time at which to plot the electric field:

$t_{plot} := 0$ Time at which to plot the electric field as a function of z (s).

Next choose the number of points to plot E_x:

$npts := 120$ Number of points to plot in z.

$z_{start} := -\frac{6\cdot\pi}{\beta_1}$ $z_{end} := \frac{6\cdot\pi}{\beta_2}$ z starting and ending points (m).

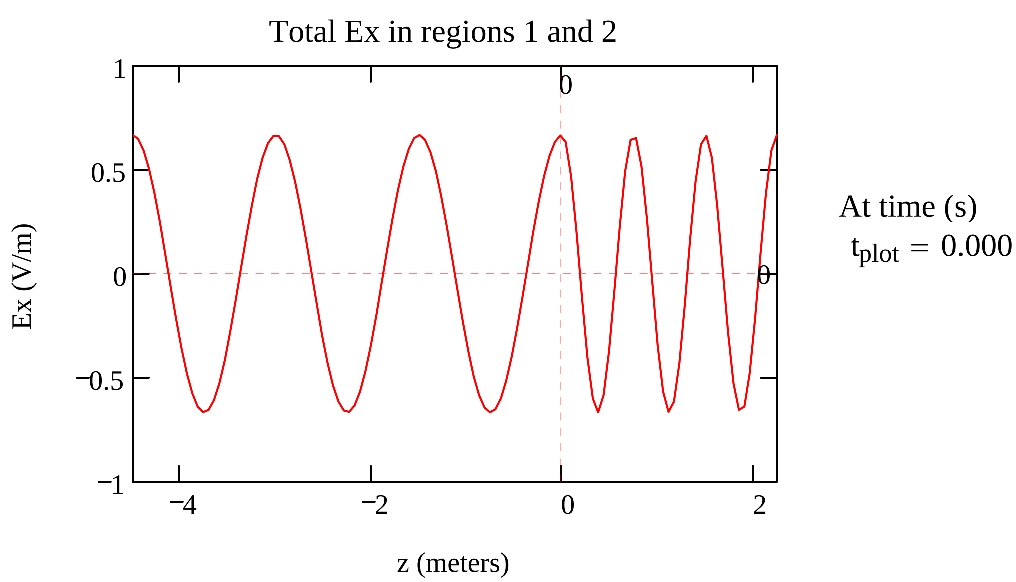
Construct a list of z_p points at which to plot E_x:

$i := 0 .. npts - 1$ $z_{p_i} := z_{start} + i\cdot\frac{z_{end} - z_{start}}{npts - 1}$

Compute the electric field at these points at time = t_{plot}:

$e_{x_i} := E_x\left(z_{p_i}, t_{plot}\right)$

Now plot the electric field E_x along the z axis at time = $t_{plot} = 0$ (s):



The interface between these two half spaces is located at z = 0. To the left of this interface is region 1 and to the right is region 2.

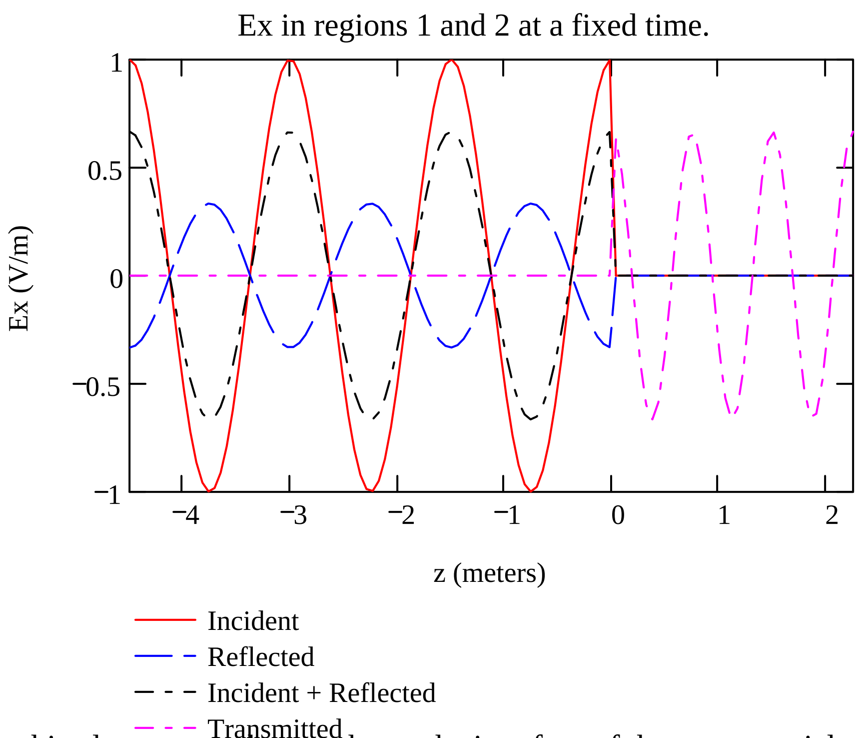
If you enter the same material parameters and frequency as in Example 6.8 of the text, you will observe in this plot that the spacing between the peaks of the wave (which is the **wavelength**) is larger in region 1 than in region 2. This is the expected result since glass has a larger relative permittivity than free space (i.e., glass is electromagnetically "denser" than vacuum) and, consequently, the wavelength is smaller in the glass material than in vacuum as we see here.

The plot above shows the variation of the *total* field throughout space. Now we will examine the components of the total waves in both regions. Specifically, the total field in region 1 (vacuum) is the sum of an incident and reflected wave, while in the region 2 the total field is onl the transmitted wave.

Compute the incident, reflected and transmitted fields along the z axis:

$$e_{ix_i} := E_{ix}\left(z_{p_i}, t_{plot}\right) \qquad e_{rx_i} := E_{rx}\left(z_{p_i}, t_{plot}\right) \qquad e_{tx_i} := E_{tx}\left(z_{p_i}, t_{plot}\right)$$

Plot the incident (region 1), reflected (region 1), incident + reflected (region 1) and the transmitted (region 2) E_x fields:



At time (s)
$t_{plot} = 0.000$

In this plot we can observe that at the interface of the two materials (at z = 0), the incident, reflected and transmitted fields are separately not continuous. The incident and reflected fields are defined only in region 1 (z < 0) while the transmitted field is defined only in region 2 (z > 0)

As we know, there is a **boundary condition** that this electric field must satisfy at the interface a z = 0. Specifically, the tangential components of the electric field must be continuous. Since E_x i strictly tangential to this interface, it must be continuous at z = 0. However, this boundary condition pertains to the *total* electric field on either side of the interface. So it is the sum of the incident and reflected field just to the left of this interface that is equal to the transmitted field ju to the right of the interface which we can observe to be satisfied in this plot.

Animated plot of E_x versus z

The next plot we will construct is the total electric field along the z axis and animated in time. Choose the number of time instances at which to plot the electric field wave:

$n_{pts} := 20$ Number of points to plot in time.

$t_{start} := 0 \qquad t_{end} := T_p$ Time to start and end plot (s).

Define the variable time in terms of the constant FRAME:

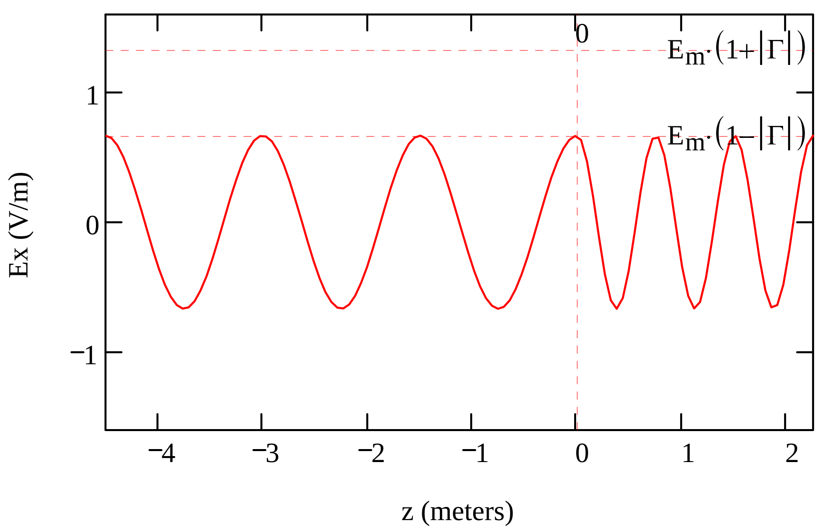
$$t_{inc} := \frac{T_p}{n_{pts}} \qquad time := t_{start} + FRAME \cdot t_{inc}$$

Define the function e_x which computes the electric field at the list of points z_p at t = time.

$$e_{x_i} := E_x\left(z_{p_i}, time\right)$$

Now generate the animation clip for the total electric field in regions 1 and 2. For best results, in the "Animate" dialog box, choose To = 19 then save the file and replay the animation in a video player that supports continuous loop playback.

Ex as a function of z and t.



Time (in periods, T_p)

$$\frac{time}{T_p} = 0.000$$

After viewing this animation, it should be clear that the waves in these two regions behave quite differently. Although the waves in both regions have a net propagation in the +z direction, we can see in this animation that the wave in region 2 (z > 0) has a constant amplitude. That is, as the wave moves towards the right, there is no change in the height of the wave. The wave in region 1 (z < 0), on the other hand, has an amplitude that is oscillating between the two levels shown by the horizontal dashed lines in the plot. These maximum and minimum amplitudes are given in Equations (133a) and (133b) in Chap. 6 of the text.

Notice in this plot that the transmitted electric field has an amplitude equal to $E_m(1 - |\Gamma|)$. Will this always be the case?

Standing wave ratio

The difference in the behavior of the fields in regions 1 and 2 is due to the fact that there exist interfering incident and reflected waves in region 1 whereas there is only a single outgoing wave in region 2. This interference (the addition and subtraction of the field values at each position and time) leads to the "pulsating" behavior we observe in region 1.

This interference produces a field in region 1 that is commonly called a *standing wave*. Actually, to call the field in region 1 a "standing" wave is a bit of a misnomer. We can clearly see in the above animation that the wave in region 1 has a net motion towards the right. This indicates that the net **time-average power flow** is in the +z direction. Only in the case of a **perfect conducting half space** (or plane) in region 2 would the wave motion in region 1 disappear altogether. In such an instance, the field would then be called a *perfect standing wave*.

We will compute the standing wave ratio, S, by using Equation (134) from Chap. 6 of the text:

$$S := \frac{1 + |\Gamma|}{1 - |\Gamma|} \qquad S = 2.000$$

Standing wave ratio in region 1 (z < 0).

The quantities in the numerator and denominator of this expression – when multiplied by E_m – are shown as horizontal lines in the above animation plot.

After viewing the animation clip and considering the equation for S, can you develop a precise word definition for the standing wave ratio?

End of worksheet.

