

Lecture 28: Poynting's Theorem. Power Flow and Plane Waves.

A propagating electromagnetic (EM) wave **carries energy** with it. Physically this makes sense to us when we listen to the radio or talk on a cell phone. These types of wireless communications are possible because EM waves carry energy.

In these examples, some of this EM energy is used to **oscillate electrons** in the metal parts of the receiving antenna of our radio or cell phone, which ultimately results in wireless communications.

There is a precise mathematical definition of the time rate of energy flow (i.e., power flow) for EM waves. Before getting to this, we first need to digress briefly to first discuss Poynting's theorem.

Poynting's Theorem

Poynting's theorem is a hugely important mathematical statement in electromagnetics that concerns the **flow of power through space**. We'll derive it now for time-domain fields.

We begin with Maxwell's curl equations

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad (1)$$

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} + \bar{J} \quad (2)$$

Next, we employ the vector calculus identity

$$\nabla \cdot (\bar{E} \times \bar{H}) = \bar{H} \cdot (\nabla \times \bar{E}) - \bar{E} \cdot (\nabla \times \bar{H}) \quad (3)$$

Substituting (1) and (2) into (3) we have:

$$\nabla \cdot (\bar{E} \times \bar{H}) = -\bar{H} \cdot \frac{\partial \bar{B}}{\partial t} - \bar{E} \cdot \frac{\partial \bar{D}}{\partial t} - \bar{E} \cdot \bar{J} \quad (4)$$

Using the constitutive equations $\bar{B} = \mu \bar{H}$, $\bar{D} = \varepsilon \bar{E}$, and $\bar{J} = \sigma \bar{E}$ in (4) gives

$$\nabla \cdot (\bar{E} \times \bar{H}) = -\mu \bar{H} \cdot \frac{\partial \bar{H}}{\partial t} - \varepsilon \bar{E} \cdot \frac{\partial \bar{E}}{\partial t} - \sigma \bar{E} \cdot \bar{E} \quad (5)$$

assuming $\mu \neq f(t)$ and $\varepsilon \neq f(t)$.

In this last equation (5), we can rewrite $\bar{H} \cdot \partial \bar{H} / \partial t$ as

$$\bar{H} \cdot \frac{\partial \bar{H}}{\partial t} = \frac{1}{2} \frac{\partial (\bar{H} \cdot \bar{H})}{\partial t} = \frac{1}{2} \frac{\partial |\bar{H}|^2}{\partial t}$$

thinking of the chain rule of differentiation. Consequently, using this result and a similar one for the $\bar{E} \cdot \partial \bar{E} / \partial t$ term, (5) becomes

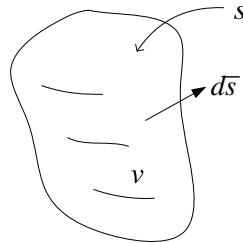
$$\nabla \cdot (\bar{E} \times \bar{H}) = -\frac{\partial}{\partial t} \left(\frac{1}{2} \mu |\bar{H}|^2 + \frac{1}{2} \varepsilon |\bar{E}|^2 \right) - \sigma |\bar{E}|^2 \quad (6)$$

This is the **point form** of what is called **Poynting's theorem**.

Lastly, we integrate this point form equation (6) throughout a volume v bounded by the closed surface s :

$$\int_v \nabla \cdot (\bar{E} \times \bar{H}) dv = - \int_v \frac{\partial}{\partial t} \left(\frac{1}{2} \mu |\bar{H}|^2 + \frac{1}{2} \varepsilon |\bar{E}|^2 \right) dv - \int_v \sigma |\bar{E}|^2 dv$$

where v , s , and $d\bar{s}$ are related as



Applying the divergence theorem to the LHS of this last equation gives

$$\oint_{s(v)} (\bar{E} \times \bar{H}) \cdot d\bar{s} = - \frac{\partial}{\partial t} \int_{v(s)} \left(\frac{1}{2} \mu |\bar{H}|^2 + \frac{1}{2} \varepsilon |\bar{E}|^2 \right) dv - \int_{v(s)} \sigma |\bar{E}|^2 dv \quad (7)$$

where in moving $\partial/\partial t$ outside of the integral we're assuming that v is not a function of time. This result is called the **integral form of Poynting's theorem**.

Discussion of Poynting's Theorem

While (7) is a hugely important theorem in electromagnetics, we are not actually going to make any calculations with it in this course. Rather, we are interested here in elucidating the physical meaning of the term $\bar{E} \times \bar{H}$ in the LHS of (7).

To understand this physical significance of the LHS of (7), we'll begin by looking at the RHS, which has elements you've seen before in EE 381 *Electric and Magnetic Fields*.

In particular, the first and second terms in the RHS of (7) are the **time rates-of-change of the stored energy in the magnetic and electric fields inside ν** . The third term is the **Ohmic power dissipated in ν** due to the flow of conduction current.

We can now interpret the RHS of (7) as the **decrease** (because of the negative sign) **in the magnetic and electric power stored in ν** , and **further reduced by the Ohmic power dissipated in ν** .

OK, so now here is the payoff: By the conservation of energy law, all of this represented by the RHS of (7) must equal the power leaving the volume through the bounding surface s .

Consequently, the quantity $\bar{\mathbf{E}} \times \bar{\mathbf{H}}$ in the LHS of (7) is a vector that must represent the **power flow of the EM field leaving the volume ν per unit area**. That is, $\bar{\mathbf{E}} \times \bar{\mathbf{H}}$ is the power flow density of electromagnetic fields.

We define this vector

$$\bar{\mathbf{S}}(t) = \bar{\mathbf{E}}(t) \times \bar{\mathbf{H}}(t) \quad [\text{W/m}^2] \quad (8)$$

(or $\bar{\mathbf{P}} = \bar{\mathbf{E}} \times \bar{\mathbf{H}}$) as the **instantaneous Poynting vector**.

Multiplying (7) by a negative sign leads to a slightly different but equivalent way to interpret (7):

$$-\oint_{s(\nu)} \bar{\mathbf{S}} \cdot d\bar{\mathbf{s}} = \frac{\partial}{\partial t} (W_m + W_e) + P_\sigma$$

The LHS is the power flow into s . The first term in the RHS is the increase in the stored power in the $\bar{\mathbf{H}}$ and $\bar{\mathbf{E}}$ fields in ν , while the second term is the increase in the Ohmic power dissipated in ν .

Power Flow for UPWs

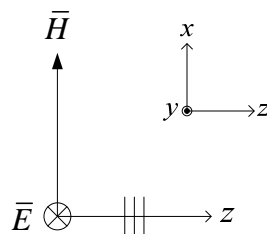
We will now apply this Poynting vector concept to uniform plane waves. In Example N26.1, we found for a certain UPW that

$$\bar{\mathbf{E}}(z, t) = -\hat{\mathbf{a}}_y 43.501 \cos(6\pi \times 10^8 t - 21.780z) \text{ V/m} \quad (9)$$

for the given magnetic field

$$\bar{\mathbf{H}}(z, t) = \hat{\mathbf{a}}_x 0.1 \cos(6\pi \times 10^8 t - 21.780z) \text{ A/m} \quad (10)$$

This UPW is propagating in the $+z$ direction:



The instantaneous Poynting vector associated with this UPW using (8) is then

$$\begin{aligned}\bar{S}(z,t) &= \bar{E}(z,t) \times \bar{H}(z,t) \\ &= (-\hat{a}_y \times \hat{a}_x) 4.350 \cos^2(6\pi \times 10^8 t - 21.780z) \text{ W/m}^2\end{aligned}$$

such that

$$\bar{S}(z,t) = \hat{a}_z 4.350 \cos^2(6\pi \times 10^8 t - 21.780z) \text{ W/m}^2 \quad (11)$$

The direction of this $\bar{S}(z,t)$ is \hat{a}_z . This indicates that the **flow of power of this UPW in the same direction as the wave propagation**: in the \hat{a}_z direction.

Time Average Power Flow

Notice in (11) that while $\bar{S}(z,t)$ oscillates in time, it has a **non-zero time average value**. (As an aside, this is one of the reasons why $\bar{S}(z)$ is not a phasor quantity.)

In particular, using the trigonometric identity $\cos^2 x = 1/2 + 1/2 \cos(2x)$, then (11) becomes

$$\bar{S}(z,t) = \hat{a}_z 2.175 \left\{ 1 + \cos \left[2(6\pi \times 10^8 t - 21.780z) \right] \right\} \text{ W/m}^2 \quad (12)$$

The second term in the RHS oscillates in time (at twice the frequency of \bar{E} and \bar{H}) and has a zero time average value, while the first term is constant and does not vary with time.

Consequently, the time average value of this Poynting vector in (12) is

$$\bar{S}_{av}(z) \equiv \frac{1}{T} \int_{t_0}^{t_0+T} \bar{S}(z,t) dt = \hat{a}_z 2.175 \text{ W/m}^2 \quad (13)$$

This UPW – on time average – carries or transfers power in the direction that the wave is propagating.

Sinusoidal Steady State Time Average Power Flow

It turns out that there is another way to calculate this time average Poynting vector for sinusoidal steady state fields, and to calculate it directly from phasor fields.

We derive this expression beginning with (8) and writing $\bar{E}(t)$ and $\bar{H}(t)$ in terms of their phasor forms

$$\begin{aligned} \bar{S}(t) &= \bar{E}(t) \times \bar{H}(t) \\ &= \text{Re}[\bar{E}e^{j\omega t}] \times \text{Re}[\bar{H}e^{j\omega t}] \end{aligned} \quad (14)$$

For evaluating (14), note that

$$\text{Re}[\bar{E}e^{j\omega t}] \times \text{Re}[\bar{H}e^{j\omega t}] \neq \text{Re}[\bar{E} \times \bar{H}e^{j2\omega t}]$$

Rather, we can employ

$$\text{Re}[\bar{A}] = \frac{1}{2}(\bar{A} + \bar{A}^*) \quad (15)$$

in (14) to give

$$\begin{aligned}\bar{S}(t) &= \frac{1}{2}(\bar{E}e^{j\omega t} + \bar{E}^*e^{-j\omega t}) \times \frac{1}{2}(\bar{H}e^{j\omega t} + \bar{H}^*e^{-j\omega t}) \\ &= \frac{1}{4}(\bar{E} \times \bar{H}e^{j2\omega t} + \bar{E} \times \bar{H}^* + \bar{E}^* \times \bar{H} + \bar{E}^* \times \bar{H}^*e^{-j2\omega t})\end{aligned}$$

which we can write as

$$\bar{S}(t) = \frac{1}{4} \left[\bar{E} \times \bar{H}^* + \bar{E} \times \bar{H}e^{j2\omega t} + (\bar{E} \times \bar{H}^* + \bar{E} \times \bar{H}e^{j2\omega t})^* \right] \quad (16)$$

We can recognize the RHS as a term plus its complex conjugate. So once again using (15), (16) becomes

$$\bar{S}(t) = \frac{1}{2} \operatorname{Re} \left[\bar{E} \times \bar{H}^* + \bar{E} \times \bar{H}e^{j2\omega t} \right] \quad (17)$$

Integrating this expression over one time period, as defined in (13), we find from (17) that

$$\boxed{\bar{S}_{av} = \frac{1}{2} \operatorname{Re} \left[\bar{E} \times \bar{H}^* \right] \quad \left[\text{W/m}^2 \right]} \quad (18)$$

Using this equation, we can **compute a time averaged quantity** (\bar{S}_{av}) **directly from phasor domain quantities** (\bar{E} and \bar{H}).

For the UPW of Example N26.1, the phasor form of \bar{E} and \bar{H} are

$$\bar{E}(z) = -\hat{a}_y 43.501e^{-j21.780z} \quad \text{V/m}$$

and

$$\bar{H}(z) = \hat{a}_x 0.1e^{-j21.780z} \quad \text{A/m}$$

Using (18), the time average Poynting vector is then

$$\begin{aligned}\bar{S}_{av} &= \frac{1}{2} \operatorname{Re} \left[-\hat{a}_y 43.501 e^{-j21.780z} \times \hat{a}_x 0.1 e^{+j21.780z} \right] \\ &= \frac{1}{2} \operatorname{Re} \left[\hat{a}_z 4.350 \right] = \hat{a}_z 2.175 \text{ W/m}^2\end{aligned}\tag{19}$$

This is the same result we found in (13) using the time domain forms of \bar{E} and \bar{H} . Here in (18) we find a time averaged quantity directly from the phasor domain fields. Neat!