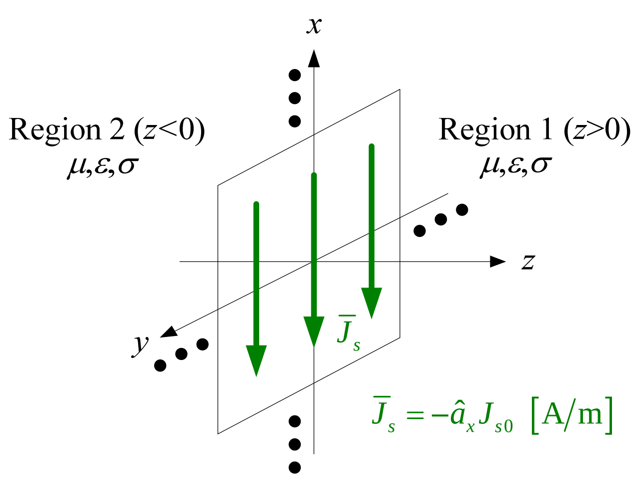
Lecture 27: Uniform Plane Waves in Lossy Materials. Skin Depth.

All real materials have loss, to one extent or another. We will now consider uniform plane waves that propagate through lossy materials. The only loss mechanism we'll consider here is that due to conduction current, $\overline{J} = \sigma \overline{E}$, everywhere in an infinite space.

So, for example, perhaps we have a current sheet $\overline{J}_s$ producing (or radiating) electromagnetic plane waves in an infinite lossy space:



Wave Equation for Lossy Media

We'll derive the wave equation for this lossy space. We begin with the curl forms of Faraday's and Ampere's laws:

$$\nabla \times \overline{E} = -j\omega\mu\overline{H} \tag{1}$$

$$\begin{aligned}\nabla \times \bar{H} &= j\omega\varepsilon\bar{E} + \bar{J} \\ &= j\omega\varepsilon\bar{E} + \sigma\bar{E} = (\sigma + j\omega\varepsilon)\bar{E}\end{aligned} \quad (2)$$

Notice that we now have a $\bar{J}$ (=$\sigma\bar{E}$) in Ampere's law (2) in contrast to a lossless media in the previous lecture.

Proceeding as we did in the previous lecture, there is no spatial variation of the current sheet and the space in the *x* or *y* directions, so we expect the $\bar{E}$ or $\bar{H}$ fields not to vary in *x* or *y* either.

Consequently, with $\partial/\partial x \to 0$ and $\partial/\partial y \to 0$, then (1) and (2) become, respectively,

$$-\hat{a}_x \frac{dE_y}{dz} + \hat{a}_y \frac{dE_x}{dz} = -j\omega\mu\bar{H} \quad (3)$$

and

$$-\hat{a}_x \frac{dH_y}{dz} + \hat{a}_y \frac{dH_x}{dz} = (\sigma + j\omega\varepsilon)\bar{E} \quad (4)$$

To form the wave equation for $\bar{E}$ in this lossy space, we first take d/dz of (3)

$$-\hat{a}_x \frac{d^2E_y}{dz^2} + \hat{a}_y \frac{d^2E_x}{dz^2} = -j\omega\mu\frac{d\bar{H}}{dz} \quad (5)$$

or separately

$\hat{a}_x$:
$$-\frac{d^2E_y}{dz^2} + j\omega\mu\frac{dH_x}{dz} = 0 \quad (6)$$

$\hat{a}_y$:
$$\frac{d^2E_x}{dz^2} + j\omega\mu\frac{dH_y}{dz} = 0 \quad (7)$$

Next, we substitute for dH_x/dz and dH_y/dz from (4) to give

$$\hat{a}_x: \qquad \frac{d^2E_y}{dz^2} - j\omega\mu(\sigma + j\omega\varepsilon)E_y = 0 \tag{8}$$

$$\hat{a}_y: \qquad \frac{d^2E_x}{dz^2} - j\omega\mu(\sigma + j\omega\varepsilon)E_x = 0 \tag{9}$$

As we learned in the previous lecture, because $\bar{J}_s$ is uniform and oriented only in the $\hat{a}_x$ direction, then the $\bar{E}$ fields produced by this current sheet will be oriented only in the $\hat{a}_x$ direction as well. From here onwards, we'll focus only on E_x and its associated $\bar{H}$ (an H_y).

From (9), we'll define the propagation constant γ such that

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) \tag{10}$$

leading from (9) to

$$\boxed{\frac{d^2E_x}{dz^2} - \gamma^2 E_x = 0} \tag{11}$$

This is the phasor wave equation for E_x in a lossy space. Notice that this equation is identical in form to the wave equation for V on a lossy transmission line in Lecture 21.

Solutions to the Wave Equation in a Lossy Space

The solutions to the wave equation (11) are

$$E_x(z) = E_0^+ e^{-\gamma z} + E_0^- e^{\gamma z} \tag{12}$$

where E_0^+ and E_0^- are complex constants. Defining the real and imaginary parts of γ as

$$\gamma \equiv \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} \text{ [m}^{-1}\text{]} \tag{13}$$

where, as for a lossy TL,

$$\alpha = \text{Re}[\gamma] = \text{attenuation constant [Np/m]} \tag{14}$$

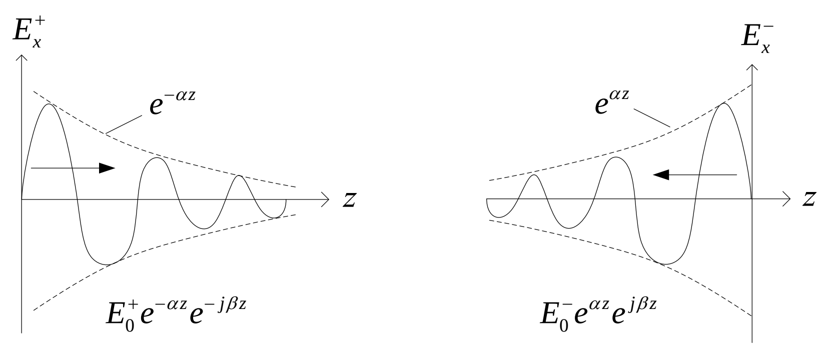
and

$$\beta = \text{Im}[\gamma] = \text{phase constant [rad/m]} \tag{15}$$

then

$$E_x(z) = E_0^+ e^{-\alpha z} e^{-j\beta z} + E_0^- e^{\alpha z} e^{j\beta z} \text{ [V/m]} \tag{16}$$

As we saw in Lecture 21, these two terms are the phasor forms of waves propagating in the +*z* and –*z* directions, respectively, with attenuation:



Associated with this time varying electric field is a corresponding magnetic field. To determine this $\bar{H}$, we could derive the wave equation for $\bar{H}$ from (1) and (2), as we did for $\bar{E}$, and then solve for $\bar{H}$.

Alternatively, and more simply, we can determine $\bar{H}$ directly from Faraday's law:

$$\bar{H} = -\frac{1}{j\omega\mu}\nabla\times\bar{E} = -\frac{1}{j\omega\mu}\begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 0 & \frac{d}{dz} \\ E_x & 0 & 0\end{vmatrix} = -\hat{a}_y\frac{1}{j\omega\mu}\frac{dE_x}{dz} \quad (17)$$

Using (12) in (17) we find that

$$\begin{aligned} H_y(z) &= -\frac{1}{j\omega\mu}(-\gamma)E_0^+e^{-\gamma z} - \frac{1}{j\omega\mu}\gamma E_0^- e^{\gamma z} \\ &= \frac{\gamma}{j\omega\mu}E_0^+e^{-\gamma z} - \frac{\gamma}{j\omega\mu}E_0^-e^{\gamma z} \end{aligned} \quad (18)$$

The factor $j\omega\mu/\gamma$ in (18) we expect to be an impedance quantity. Using (10) we find

$$\eta_c \equiv \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \quad [\Omega] \quad (19)$$

This quantity η_c is the complex intrinsic impedance of the lossy space. It is a complex quantity, just as the characteristic impedance of a lossy TL was found to be complex in Lecture 21. Notice in (19) that as $\sigma \to 0$, $\eta_c \to \eta$ as expected for a lossless space, as we saw in the previous lecture.

Using this η_c in (18), the magnetic field associated with E_x in (16) for this lossy space is

$$H_y(z) = \frac{E_0^+}{\eta_c}e^{-\alpha z}e^{-j\beta z} - \frac{E_0^-}{\eta_c}e^{\alpha z}e^{j\beta z} \ \text{[A/m]} \quad (20)$$

Discussion

From (16) and (20), notice that

- The fields do not vary spatially in planes perpendicular to the direction of propagation ($\pm z$). Even though the space is lossy, these fields are still uniform plane waves, as in the case of a lossless space.

- Ratios of perpendicular components of $\bar{E}$ and $\bar{H}$ equal $\pm\eta_c$.
  - ✓ For the +z propagating wave:
  $$\frac{E_x^+}{H_y^+} = \eta_c$$
  - ✓ For the -z propagating wave:
  $$\frac{E_x^-}{H_y^-} = -\eta_c$$

  This is a result similar to lossless spaces, but η_c is now complex for a lossy space.

- For a lossy space, the wavelength and wave speed are defined exactly the same for lossless spaces as $\lambda = 2\pi/\beta$ and $u = \omega/\beta$, respectively. However, $\beta \neq \omega\sqrt{\mu\varepsilon}$ for lossy spaces. Rather, from (15) $\beta = \operatorname{Im}[\gamma]$.

Skin Depth

We've now learned that as a plane wave propagates through a lossy space, it attenuates in amplitude as it propagates. For a UPW propagating in the +z direction, for example, from (16) and (20)

$$E_x^+ = E_0^+ e^{-\alpha z} e^{-j\beta z} \quad \text{and} \quad H_y^+ = \frac{E_0^+}{\eta_c} e^{-\alpha z} e^{-j\beta z} \tag{21}$$

The distance this electromagnetic wave must travel for the amplitude to be reduced by the factor e^{-1} is called the **skin depth**, δ, of the material.

We can derive an equation for δ beginning with the magnitude of the electric field in (21)

$$\left|E_x^+(z)\right| = \left|E_0^+\right| e^{-\alpha z} \tag{22}$$

At some arbitrary position z_0, from (22)

$$\left|E_x^+(z_0)\right| = \left|E_0^+\right| e^{-\alpha z_0} \tag{23}$$

while at some position δ meters further away

$$\left|E_x^+(z_0+\delta)\right| = \left|E_0^+\right| e^{-\alpha(z_0+\delta)} \tag{24}$$

At this position, the field will decay by the factor e^{-1}, because of the definition of δ.

Forming the ratio of (24) and (23)

$$\frac{\left|E_x^+(z_0+\delta)\right|}{\left|E_x^+(z_0)\right|}=\frac{\left|E_0^+\right|e^{-\alpha z_0}e^{-\alpha\delta}}{\left|E_0^+\right|e^{-\alpha z_0}}=e^{-\alpha\delta}$$

For this ratio to equal e^{-1} requires

$$\boxed{\delta=\frac{1}{\alpha}\quad[\text{m}]}\qquad(25)$$

As an example, consider copper which has an electrical conductivity $\sigma=5.8\times10^7$ S/m, $\varepsilon\approx\varepsilon_0$, and $\mu\approx\mu_0$. Using (13) to calculate γ, then α from (14), and using (25):

f	$\delta=1/\alpha$
60 Hz	8530 μm
1 MHz	66.1 μm
10 MHz	20.9 μm
100 MHz	6.6 μm
1 GHz	2.09 μm

This skin depth of an electromagnetic wave in a lossy space is directly related to the skin effect in a round wire that we discussed in Lecture 9.