Lecture 26: Uniform Plane Waves. Infinite Current Sheets.

Over the past weeks, we've seen that voltage and current waves can propagate along transmission lines. Examples of TLs, as we saw, included coaxial cables, twin lead, and lands on a circuit board.

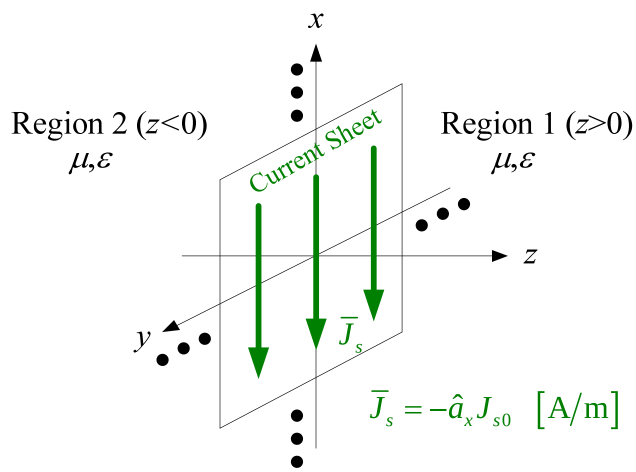
These voltage and current waves are also examples of electromagnetic (EM) waves.

There are many more examples of EM waves:

- Radio
- Radar
- Satellite communications
- Light
- Microwave ovens

The next major topic in this course is the simplest examples of EM waves propagating without a supporting structure. These simplest examples are called **uniform plane waves (UPWs)**.

Uniform plane waves are produced by infinite planar sheets of surface current density, $\bar{J}_s$. (Your text denotes a surface current density by the vector $\bar{K}$.) In our work here, we will assume only sinusoidal steady state so that $\bar{J}_s$ is a phasor.



There is no supporting structure here. We are simply assuming that the current exists as an impressed (or source) current sheet as a means to begin studying electromagnetic waves that propagate in space without a guiding structure (as they had with transmission lines).

More practical examples of finite-sized sources producing electromagnetics waves will be discussed later in the course when we talk about antennas.

Wave Equation

The task before us is to solve for the $\bar{E}$ and $\bar{H}$ produced everywhere in space by this electric current sheet.

We will begin this solution process with the phasor forms of Maxwell's curl equations

$$\nabla \times \bar{E} = -j\omega\bar{B} = -j\omega\mu\bar{H} \tag{1}$$

$$\nabla \times \bar{H} = j\omega \bar{D} = j\omega\varepsilon \bar{E} \qquad (2)$$

Since there is no variation in the source (or space) in the x and y directions, we also would not expect any variation of $\bar{E}$ and $\bar{H}$ in x and y. Consequently, $\partial/\partial x \to 0$ and $\partial/\partial y \to 0$.

Therefore, (1) becomes

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 0 & \dfrac{d}{dz} \\ E_x & E_y & E_z \end{vmatrix} = -j\omega\mu\bar{H}$$

or

$$-\hat{a}_x \frac{dE_y}{dz} + \hat{a}_y \frac{dE_x}{dz} = -j\omega\mu\bar{H} \qquad (3)$$

Similarly for (2) we find

$$-\hat{a}_x \frac{dH_y}{dz} + \hat{a}_y \frac{dH_x}{dz} = j\omega\varepsilon\bar{E} \qquad (4)$$

We wish to determine separate differential equations for E_x alone and E_y alone from (3) and (4).

To do this, we take d/dz of (3) and substitute (4) giving

$\hat{a}_x$:
$$\boxed{\frac{d^2E_y}{dz^2} + \omega^2\mu\varepsilon E_y = 0} \qquad (5)$$

$\hat{a}_y$:
$$\boxed{\frac{d^2E_x}{dz^2} + \omega^2\mu\varepsilon E_x = 0} \qquad (6)$$

We can recognize (5) and (6) as **phasor wave equations** for E_x and E_y. They are identical in form to the voltage and current phasor wave equations for TLs we derived in Lecture 17.

Defining the **phase constant** (also called the **wavenumber**) β as

$$\boxed{\beta = \omega\sqrt{\mu\varepsilon} \text{ [rad/m]}} \tag{7}$$

then the **solutions** to (5) and (6) are

$$E_y(z) = C_1 e^{-j\beta z} + C_2 e^{+j\beta z} \tag{8}$$

$$E_x(z) = C_3 e^{-j\beta z} + C_4 e^{+j\beta z} \tag{9}$$

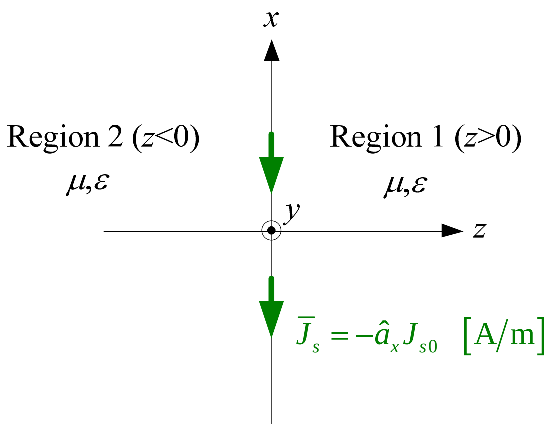
C_i, $i = 1,\ldots,4$ are (complex) constants that are evaluated by applying the boundary conditions presented by a particular problem.

As we've seen before with phasor voltages and currents on TLs, the first terms in (8) and (9) are the phasor representations of electric field **waves** propagating in the +*z* direction. Conversely, the second terms represent **waves** propagating in the –*z* direction.

Consequently, this current sheet is producing electromagnetic waves! These EM waves propagate in space without any supporting structure, such as conductors in a coaxial cable, for example.

EM Waves Produced by the Current Sheet

Returning to the current sheet problem:



In general, E_x and E_y in **each** region 1 and 2 will have the form (8) and (9).

It can be shown that a uniform $\bar{J}_s$ directed only in $\hat{a}_x$ produces $\bar{E}$ **only** in the same $\hat{a}_x$ direction. Therefore, there will only exist an E_x (and no E_y) in both regions 1 and 2 according to (9) as

Region 1: $$E_{x1} = Ae^{-j\beta z} + Be^{+j\beta z} \tag{10}$$

Region 2: $$E_{x2} = Ce^{-j\beta z} + De^{+j\beta z} \tag{11}$$

In both of these regions, we would expect only "outgoing" waves; that is, waves that travel away from the current sheet. This means that

Region 1: $E_{x1} \propto e^{-j\beta z}$ $\quad z \geq 0$

Region 2: $E_{x2} \propto e^{+j\beta z}$ $\quad z \leq 0$

Therefore, $B = C = 0$ in (10) and (11).

Also associated with these electric fields are **magnetic fields** $\bar{H}$ that we can determine from these electric fields using Maxwell's equations. In particular, using Faraday's law $\nabla \times \bar{E} = -j\omega\mu\bar{H}$ we find

$$\bar{H} = \frac{-1}{j\omega\mu}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 0 & \frac{d}{dz} \\ E_x & 0 & 0 \end{vmatrix} = \frac{-1}{j\omega\mu}\left[-\hat{a}_y\left(-\frac{dE_x}{dz}\right)\right] = \hat{a}_y \frac{j}{\omega\mu}\frac{dE_x}{dz} \quad (12)$$

Using (10) and (11) in (12) with $B = C = 0$, we can determine the magnetic fields as

Region 1: $$H_{y1} = \frac{j}{\omega\mu}\frac{dE_{x1}}{dz} = \frac{jA}{\omega\mu}(-j\beta)e^{-j\beta z} = \frac{\beta}{\omega\mu}Ae^{-j\beta z} \quad (13)$$

Region 2: $$H_{y2} = \frac{j}{\omega\mu}\frac{dE_{x2}}{dz} = \frac{jD}{\omega\mu}(j\beta)e^{+j\beta z} = -\frac{\beta}{\omega\mu}De^{+j\beta z} \quad (14)$$

To determine the remaining constants A and D, we apply the two boundary conditions for the electric current sheet at $z = 0$ (see Lecture 6):

$$\boxed{\hat{a}_z \times \left(\bar{E}_1 - \bar{E}_2\right) = 0\Big|_{z=0}} \quad (15)$$

$$\boxed{\hat{a}_z \times \left(\bar{H}_1 - \bar{H}_2\right) = \bar{J}_s\Big|_{z=0}} \quad (16)$$

Substituting (10) and (11) into (15) (with $B = C = 0$) we find

$$A = D \quad (17)$$

while substituting (13), (14), and (17) into (16) we find

$$A = D = \frac{\omega\mu}{2\beta} J_{s0} \tag{18}$$

To finish this solution, we substitute (18) into (10), (11), (13), and (14) giving

$$\boxed{\bar{E}_1 = \hat{a}_x \frac{\omega\mu}{2\beta} J_{s0} e^{-j\beta z} \quad [\text{V/m}] \qquad z \ge 0} \tag{19}$$

$$\boxed{\bar{E}_2 = \hat{a}_x \frac{\omega\mu}{2\beta} J_{s0} e^{+j\beta z} \quad [\text{V/m}] \qquad z \le 0} \tag{20}$$

$$\boxed{\bar{H}_1 = \hat{a}_y \frac{J_{s0}}{2} e^{-j\beta z} \qquad [\text{A/m}] \qquad z > 0} \tag{21}$$

$$\boxed{\bar{H}_2 = -\hat{a}_y \frac{J_{s0}}{2} e^{+j\beta z} \quad [\text{A/m}] \qquad z < 0} \tag{22}$$

We have successfully solved for the EM fields produced everywhere in space by this current sheet. Equations (19)–(22) are the complete solution for the EM waves produced by the uniform electric current sheet $\bar{J}_s = -\hat{a}_x J_{s0}$ A/m at $z = 0$.

Discussion

- From (19) and (20)

$$\frac{\omega\mu}{\beta} \underbrace{=}_{(7)} \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} \equiv \eta \quad [\Omega] \tag{23}$$

  η is called the **intrinsic impedance** of the medium with units of Ω. For vacuum, $\eta_0 = \sqrt{\mu_0 / \varepsilon_0} = 376.73037 (\approx 120\pi)\ \Omega$.

- From (19) and (21) as well as (20) and (22)
  - ✓ $\bar{E} \perp \bar{H}$.
  - ✓ Both $\bar{E}$ and $\bar{H}$ are perpendicular to z, which is the direction of propagation.
  - ✓ The direction of the cross product $\bar{E} \times \bar{H}$ is also the direction of propagation.

- The ratio of E_x to H_y in region 1 is

$$\frac{E_{x1}}{H_{y1}} \underbrace{=}_{(19),(21)} \frac{\frac{\omega\mu}{2\beta} J_{s0} e^{-j\beta z}}{\frac{J_{s0}}{2} e^{-j\beta z}} = \frac{\omega\mu}{\beta} \underbrace{=}_{(23)} \eta$$

while in region 2

$$\frac{E_{x2}}{H_{y2}} = -\eta$$

This is analogous to TLs where

$$\frac{V^+}{I^+} = Z_0 \text{ and } \frac{V^-}{I^-} = -Z_0 .$$

- In planes perpendicular to the directions of propagation $(\pm z)$ the $\bar{E}$ and $\bar{H}$ fields are constant – in both magnitude and phase. That is, the fields are **uniform**.

  Because of this property, these types of EM fields are called **uniform plane waves (UPWs)**.

- As with TLs

✓ Wavelength = $\boxed{\lambda = \frac{2\pi}{\beta} \text{ [m]}}$.

✓ Wave speed = $\boxed{u = \frac{\omega}{\beta} \text{ [m/s]}}$.

- How fast does an EM wave travel in vacuum?

$$u = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu_0\varepsilon_0}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 2.998\times 10^8 \text{ m/s}$$

which is the speed of light in a vacuum!

Example N26.1: A UPW is propagating in a material with $\varepsilon_r = 3$ and $\mu_r = 4$. This UPW has a magnetic field given as

$$\bar{H}(z,t) = \hat{a}_x 0.1\cos\left(6\pi\times 10^8 t - 21.780z\right) \text{ A/m.}$$

Determine (a) the direction of propagation, (b) the frequency, (c) the wavenumber, (d) the wavelength, and (e) write the complete time domain expression for $\bar{E}(z,t)$.

From the given magnetic field:

$$\bar{H}(z,t) = \hat{a}_x 0.1\cos\left(\underbrace{6\pi\times 10^8}_{=\omega} t - \underbrace{21.780}_{=\beta} z\right) \text{ [A/m]}$$

(a) Because of the '-' sign, wave propagation is in the +z direction.

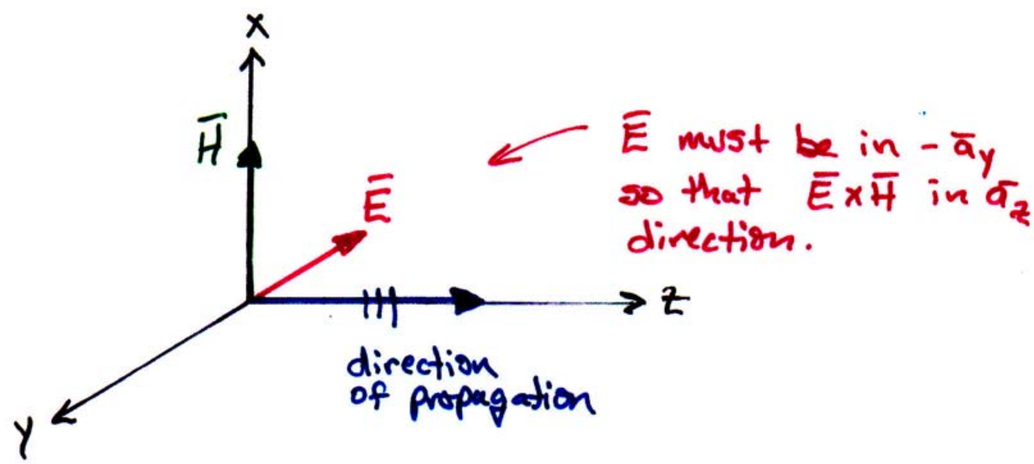
(b) $\omega = 2\pi f = 6\pi\times 10^8$ rad/s $\Rightarrow$ $f = 3\times 10^8 = 300$ MHz.

(c) Directly from the phase term, $\beta = 21.780$ rad/m.

(d) The wavelength can be computed from β as

$$\beta = \frac{2\pi}{\lambda} \quad \Rightarrow \quad \lambda = \frac{2\pi}{\beta} = 0.288 \text{ m}.$$

(e) To determine the direction of $\bar{E}$, it is very helpful to draw a sketch:



Because
$$\frac{|E_y|}{|H_x|} = \eta$$

and given the direction of $\bar{E}$ in the sketch above, then

$$\bar{E}(z,t) = -\hat{a}_y 0.1\eta \cos(\omega t - \beta z)$$

with

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}} = \eta_0\sqrt{\frac{4}{3}} = 376.73\frac{2}{\sqrt{3}} = 435.01\ \Omega$$

Therefore,

$$\bar{E}(z,t) = -\hat{a}_y 43.501\cos\left(6\pi\times 10^8 t - 21.780z\right) \text{ V/m}$$