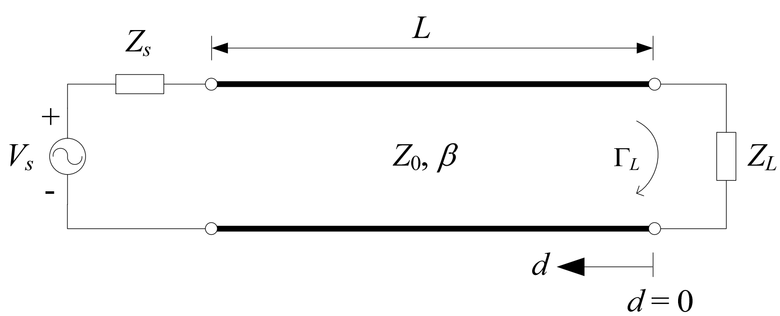
Lecture 22: Smith Chart.

The Smith chart is a very useful graphical tool for the analysis of TLs. It was developed by Phillip H. Smith in the 1930s.

The Smith chart remains a very useful tool today primarily to visualize the results of TL analysis, oftentimes combined with computer analysis and visualization.

The development of the Smith chart is based on the **normalized TL impedance** $z \equiv Z(z)/Z_0$. To avoid confusion between the lower case z's (one being normalized impedance and the other the position on the TL), we will define a new positional variable $d \equiv -z$:



There is nothing fundamentally important about this change of positional variable. It is done only for convenience.

Substituting $d = -z$ in (4) and (5) from Lecture 20, the voltage and current anywhere on the TL in terms of d are then

$$V(d) = V_o^+ e^{j\beta d}\left[1 + \Gamma(d)\right] \tag{1}$$

and
$$I(d)=\frac{V_o^+}{Z_0}e^{j\beta d}\left[1-\Gamma(d)\right] \tag{2}$$
where the generalized reflection coefficient now reads
$$\Gamma(d)=\Gamma_L e^{-j2\beta d} \tag{3}$$

So, with this new coordinate variable *d* the normalized TL impedance anywhere on the TL can be expressed in terms of the generalized reflection coefficient $\Gamma(d)$ as
$$z(d)\equiv\frac{Z(d)}{Z_0}=\frac{1+\Gamma(d)}{1-\Gamma(d)} \tag{4}$$
where $Z(d)=V(d)/I(d)$ is the TL impedance at position *d*.

This very simple equation (4) is the basis for the Smith chart. It shows that the normalized impedance at some position on the TL is uniquely related to the generalized reflection coefficient at that same point, and *vice versa*.

The real and imaginary parts of $\Gamma(d)$ will be defined as
$$\Gamma(d)\equiv p+jq \tag{5}$$
Substituting this definition into (4) gives
$$z(d)=\frac{1+(p+jq)}{1-(p+jq)} \tag{6}$$

Next, we will define $z(d)\equiv r+jx$ and separate (6) into its real and imaginary parts

$$z(d) \equiv r + jx = \frac{1+(p+jq)}{1-(p+jq)} \cdot \frac{1-(p+jq)^*}{1-(p+jq)^*}$$

$$= \frac{1+j2q-\left(p^2+q^2\right)}{1-2p+p^2+q^2} = \frac{1-\left(p^2+q^2\right)+j2q}{(p-1)^2+q^2}$$

Equating the real and imaginary parts of this equation gives

$$r = \frac{1-\left(p^2+q^2\right)}{(p-1)^2+q^2} \quad \text{and} \quad x = \frac{2q}{(p-1)^2+q^2}$$

Rearranging both of these leads us to the final two equations

$$\left(p - \frac{r}{1+r}\right)^2 + q^2 = \left(\frac{1}{1+r}\right)^2 \tag{7}$$

and

$$(p-1)^2 + \left(q - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2 \tag{8}$$

Construction of the Smith Chart

We will use (7) and (8) to construct the Smith chart.

Definition: The Smith chart is a graphical plot of normalized TL resistance and reactance functions drawn in the complex, generalized reflection coefficient [$\Gamma(d) \equiv p + jq$] plane.

To construct the Smith chart from (7) and (8), first notice that in the *p*-*q* plane:

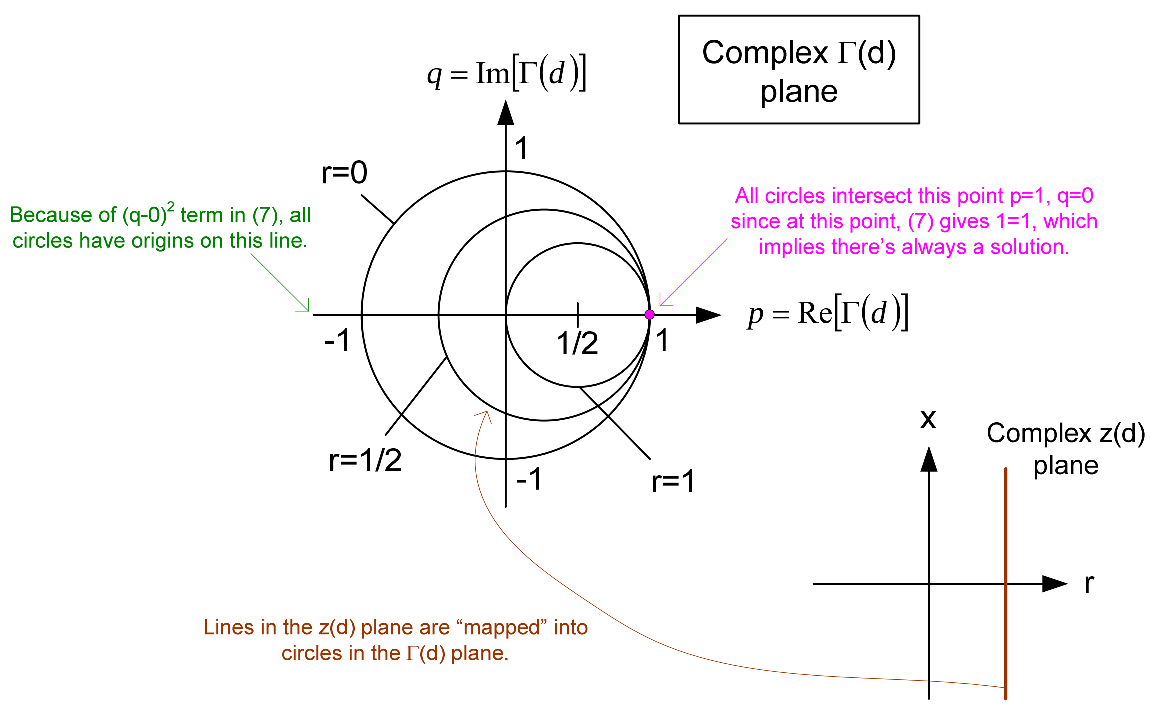
1. Equation (7) has only r as a parameter and (8) has only x as a parameter.
2. Both (7) and (8) are equations for families of circles.

Consequently, we can plot (7) and (8) in the *p*-*q* plane while keeping either r or x constant, as appropriate.

Plot (7) in the *p*-*q* plane:

- For $r=0$: $p^2+q^2=1^2$
- For $r=1$: $\left(p-\frac{1}{2}\right)^2+q^2=\left(\frac{1}{2}\right)^2$
- For $r=\frac{1}{2}$: $\left(p-\frac{1}{3}\right)^2+q^2=\left(\frac{2}{3}\right)^2$

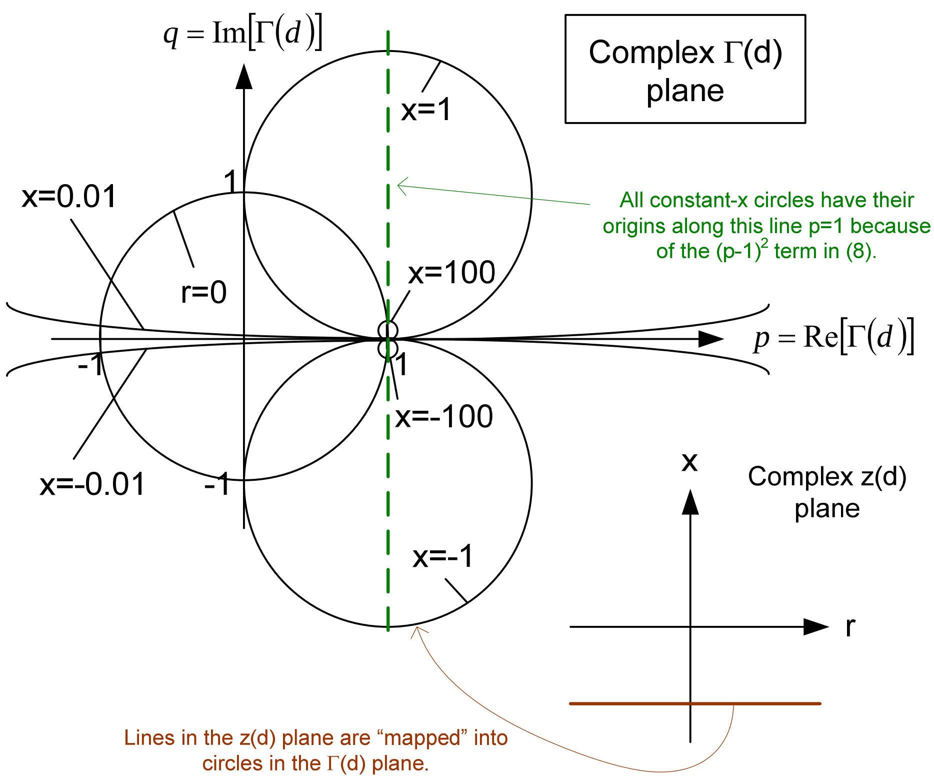
Plots of these curves in the *p*-*q* plane are:



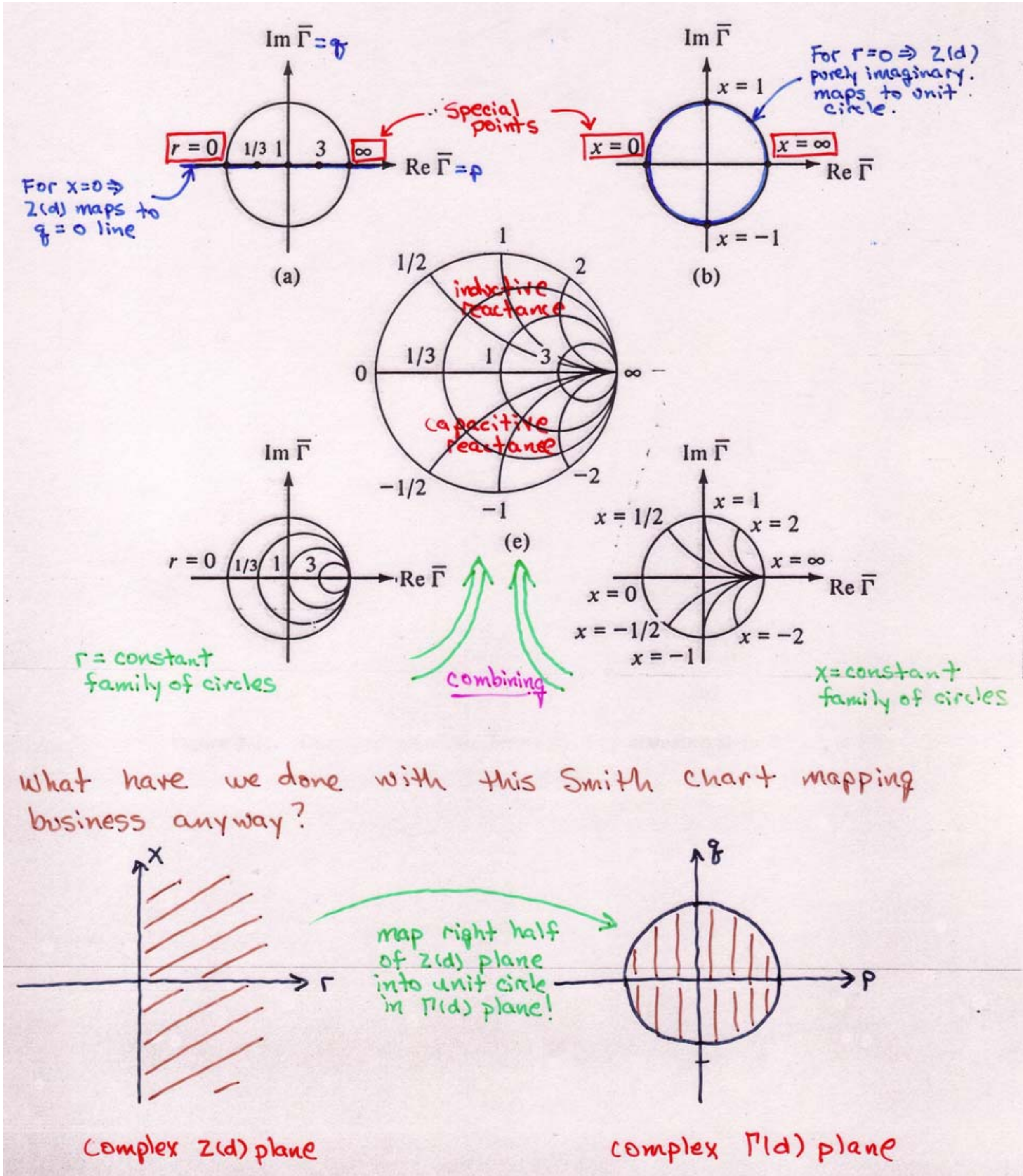
Plot (8) in the *p-q* plane:

- For $x=1$: $(p-1)^2+(q-1)^2=1^2$
- For $x=-1$: $(p-1)^2+(q+1)^2=(-1)^2$
- For $x=100$: $(p-1)^2+\left(q-\frac{1}{100}\right)^2=\left(\frac{1}{100}\right)^2$
- For $x=\frac{1}{100}$: $(p-1)^2+(q-100)^2=100^2$

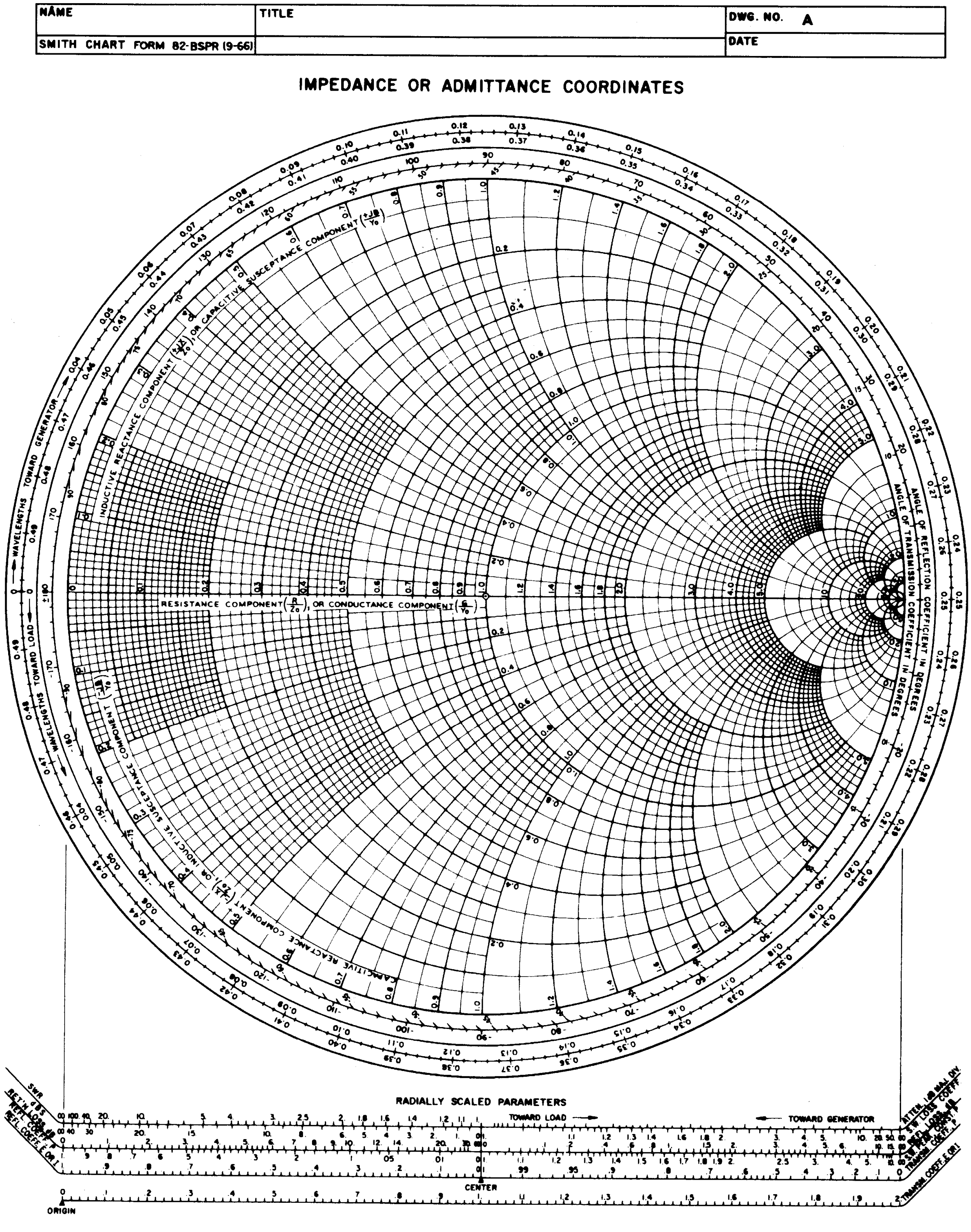
Plots of these curves in the *p-q* plane are:



Combining both of these curves (or "mappings") gives what is called the Smith chart:



The Smith chart is available in printed form from commercial suppliers. An example is shown on the next page. Note the scales at the bottom of the page.


NAME
TITLE
DWG. NO. A
SMITH CHART FORM 82-BSPR (9-66)
DATE
IMPEDANCE OR ADMITTANCE COORDINATES
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE COMPONENT (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE COMPONENT (-jB/Yo)
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
ANGLE OF TRANSMISSION COEFFICIENT IN DEGREES
RADIALLY SCALED PARAMETERS
TOWARD LOAD
TOWARD GENERATOR
CENTER
ORIGIN
A MEGA-CHART

Important Features of the Smith Chart

1. By definition $\Gamma(d) = \frac{z(d)-1}{z(d)+1} = \frac{(r+jx)-1}{(r+jx)+1}$. Then

$$|\Gamma(d)| = \frac{r+jx-1}{r+jx+1} \cdot \frac{r-jx+1}{r-jx+1} = \frac{(r-1)^2 + x^2}{(r+1)^2 + x^2}$$

From this result, we can show that if $r \geq 0$ then $|\Gamma(d)| \leq 1$. This condition is met for passive networks (i.e., no amplifiers) and lossless TLs (real Z_0).

Consequently, the standard Smith chart only shows the inside of the unit circle in the *p*-*q* plane. That is, $|\Gamma(d)| \leq 1$ which is bounded by the $r = 0$ circle located described by $p^2 + q^2 = 1$.

2. Notice that in the upper semi-circle of the Smith chart, $x \geq 0$ which is an inductive reactance (see figures on pages 5 and 6). Consequently, the generalized reflection coefficients $\Gamma(d) \equiv p + jq$ in this upper semi-circle are associated with normalized TL impedances $z(d) \equiv r + jx$ that are inductively reactive.

Conversely, in the lower semi-circle $x \leq 0$ indicating capacitive reactive TL impedance.

3. If $z(d)$ is purely real (i.e., $x = 0$) then because from p. 3

$$x = \frac{2q}{(p-1)^2 + q^2}$$

we deduce that $q = 0$ (except possibly at $p = 1$).

Consequently, purely real $z(d)$ values are mapped to $\Gamma(d)$ values on the $p = \text{Re}[\Gamma(d)]$ axis.

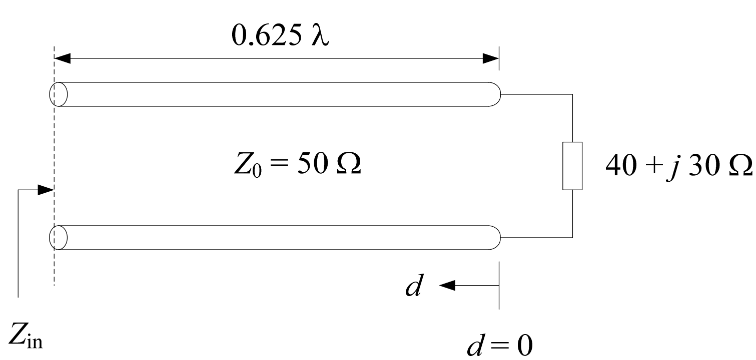
4. If $z(d)$ is purely imaginary (i.e., $r = 0$) then from (7)

$$p^2 + q^2 = 1^2$$

which is the unit circle in the *p*-*q* plane.

Consequently, purely imaginary $z(d)$ values are mapped to $\Gamma(d)$ values on the unit circle in the *p*-*q* plane.

Example N22.1: Using the Smith chart, determine the voltage reflection coefficient at the load and the input impedance of the TL below.



- $z(d=0)=\frac{Z(d=0)}{Z_0}=\frac{Z_L}{Z_0}=0.8+j0.6$ p.u.Ω. Mark this on Smith chart.

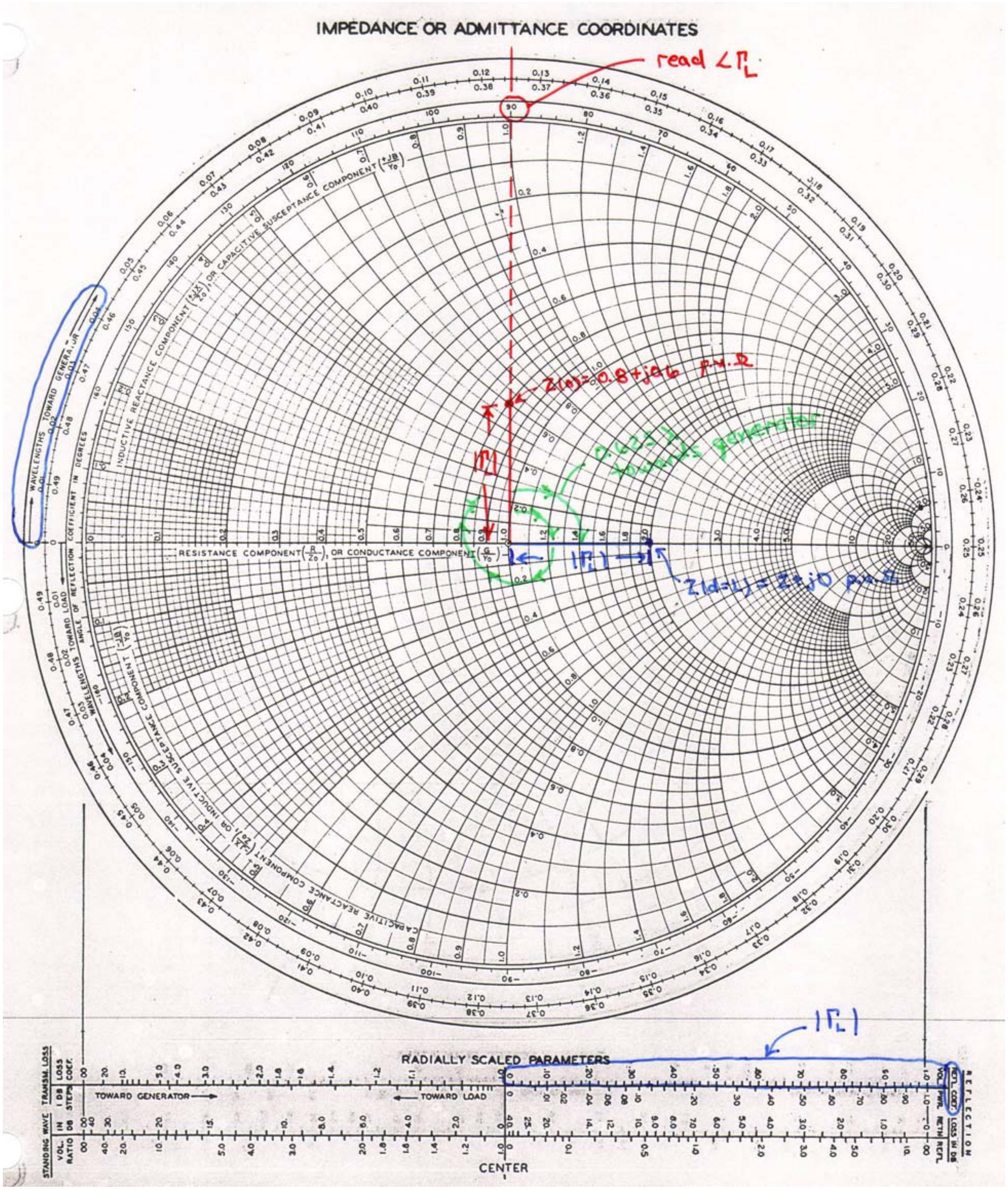
- What is Γ_L? Read this directly from the Smith chart:

$$|\Gamma_L|=0.33 \text{ and } \measuredangle\Gamma_L=90^{\circ}$$

  Therefore $\Gamma_L = j0.33$ (exact is $\frac{j}{3}$)

- For $L=0.625\lambda$, what is Z_{in}? Rotate the Γ_L vector $[=\Gamma(d=0)]$ 0.625λ towards generator. From Smith chart, read

$$z_{\text{in}}=2+j0 \quad \Rightarrow \quad Z_{\text{in}}=50\cdot 2=100\ \Omega \text{ (exact is } 100\ \Omega)$$


IMPEDANCE OR ADMITTANCE COORDINATES
read ∠Γ_L
Z(0) = 0.8 + j0.6 p.u. Ω
|Γ|
0.125 λ towards generator
|Γ_L|
Z(d=L) = 2 + j0 p.u. Ω
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
RADIALLY SCALED PARAMETERS
TOWARD GENERATOR
TOWARD LOAD
CENTER

VSWR and the Smith Chart

From (1) the voltage magnitude anywhere on the TL can be written as

$$|V(d)| = |V_o^+| |1+\Gamma(d)|$$

Using the crank diagram in Lecture 19, we found that

$$|V(d)|_{\max} = |V_o^+| (1+|\Gamma_L|)$$

and

$$|V(d)|_{\min} = |V_o^+| (1-|\Gamma_L|)$$

To help visualize this, the crank diagram has been redrawn on the Smith chart shown on p. 14.

However, when positioned along the TL at a voltage magnitude **maximum** then

$$Z(d) = \frac{V(d)}{I(d)} = \frac{V_o^+ e^{j\beta d}\left[1+\Gamma(d)\right]}{\frac{V_o^+}{Z_0} e^{j\beta d}\left[1-\Gamma(d)\right]} = Z_0 \frac{1+|\Gamma_L|}{1-|\Gamma_L|} \tag{9}$$

Using the definition of VSWR from Lecture 20

$$\text{VSWR} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|},$$

then comparing these last two equations we see that at a position along the TL where there is a voltage magnitude maximum, then

$$Z(d) = Z_0 \cdot \text{VSWR}$$

or

$$\boxed{z(d) = \text{VSWR}}$$

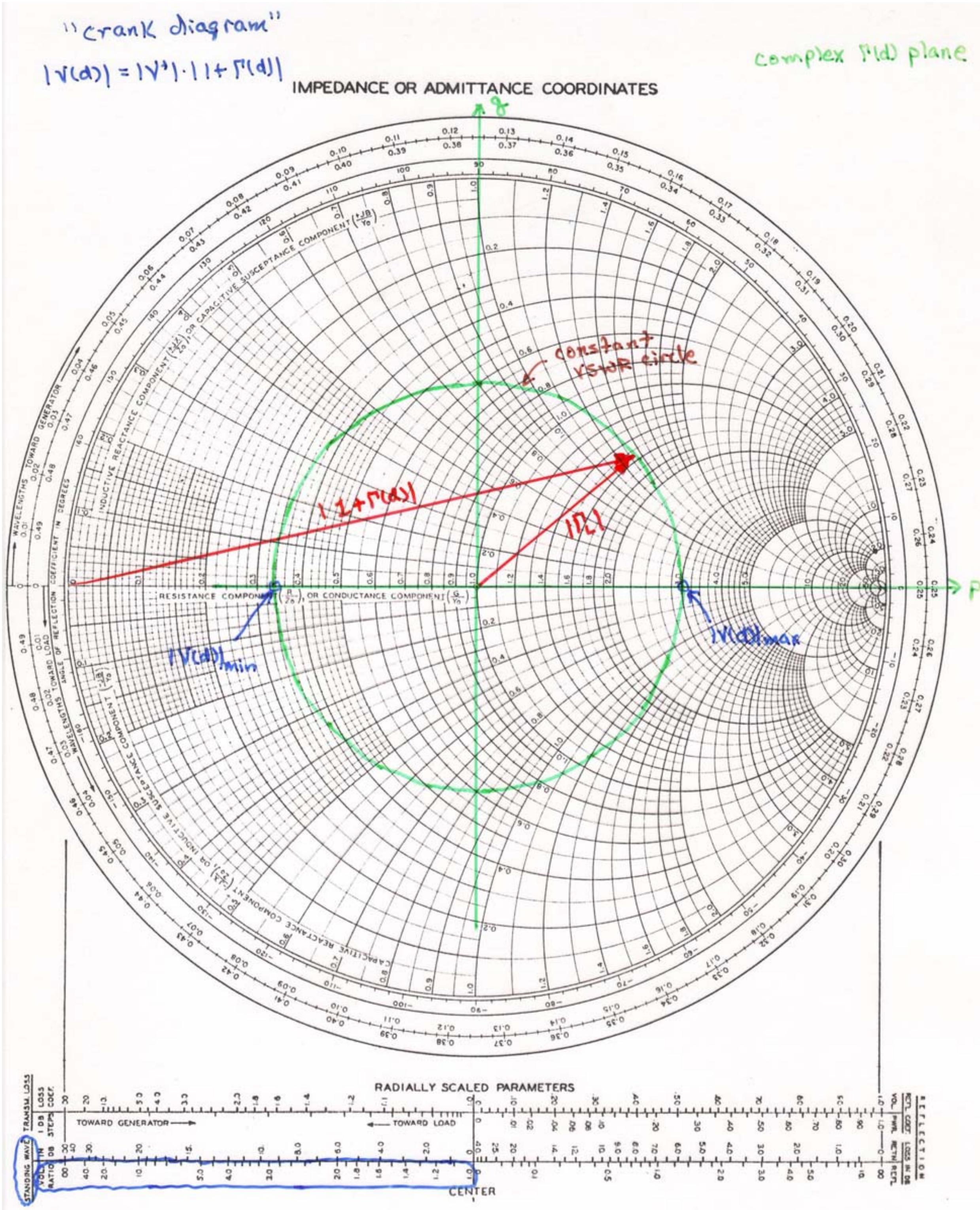
Because of this result, we can read the VSWR of a TL directly from the Smith chart.

Similarly, we can show that at a voltage magnitude minima

$$z(d) = \frac{1}{\text{VSWR}}$$

In the previous example (Example N21.1), we can read VSWR = 2 directly from the Smith chart by drawing the **constant VSWR circle**. This is the circle traced by $\Gamma(d)$ as *d* varies.

However, notice that depending on where we "stop" this rotation of $\Gamma(d)$ versus *d*, we obtain different $z(d)$ values. This happens because $\Gamma(d)$ is **not** traversing circles of constant *r* and/or *x* as *d* varies.


"crank diagram"
|V(d)| = |V+|·|1+Γ(d)|
complex Γ(d) plane
IMPEDANCE OR ADMITTANCE COORDINATES
constant VSWR circle
|1+Γ(d)|
|ΓL|
|V(d)|min
|V(d)|max
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
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RADIALLY SCALED PARAMETERS
TOWARD GENERATOR
TOWARD LOAD
STANDING WAVE
TRANSM. LOSS
REFLECTION
CENTER