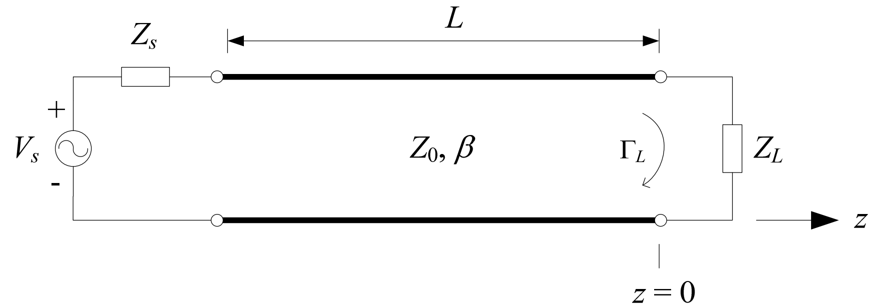
Lecture 20: Generalized Reflection Coefficient. Crank Diagram. VSWR.

As we saw in the previous lecture, for a lossless TL with an arbitrary load and this chosen coordinate system



the voltage and current on the TL can be written as

$$V(z) = V_o^+ \left(e^{-j\beta z} + \frac{V_o^-}{V_o^+} e^{+j\beta z} \right) = V_o^+ \left(e^{-j\beta z} + \Gamma_L e^{+j\beta z} \right) \quad (1)$$

and $$I(z) = \frac{V_o^+}{Z_0} \left(e^{-j\beta z} - \frac{V_o^-}{V_o^+} e^{+j\beta z} \right) = \frac{V_o^+}{Z_0} \left(e^{-j\beta z} - \Gamma_L e^{+j\beta z} \right) \quad (2)$$

In these expressions $$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

is the voltage reflection coefficient at the load.

We can "generalize" the concept of voltage reflection coefficient to be the ratio of the (complex) amplitudes of the -*z* and +*z* traveling voltage waves at any point along the TL.

That is, we define the generalized reflection coefficient $\Gamma(z)$ for a lossless TL by dividing the second term in (1) by the first term

$$\Gamma(z) \equiv \frac{V_o^+ \Gamma_L e^{+j\beta z}}{V_o^+ e^{-j\beta z}} = \Gamma_L e^{j2\beta z} \tag{3}$$

Now, substituting (3) into (1) and (2) gives

$$V(z) = V_o^+ e^{-j\beta z}\left[1 + \Gamma(z)\right] \tag{4}$$

and

$$I(z) = \frac{V_o^+}{Z_0} e^{-j\beta z}\left[1 - \Gamma(z)\right] \tag{5}$$

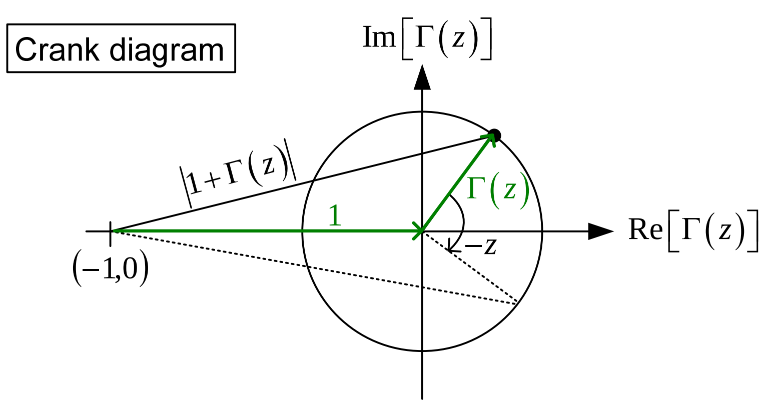
It is worthwhile to memorize (3)–(5). These equations are the foundation upon which you can understand and solve sinusoidal steady state TL problems.

Crank Diagram

Taking the magnitude of (4) gives

$$|V(z)| = |V_o^+| \, |1 + \Gamma(z)| \tag{6}$$

In the "complex $\Gamma(z)$" plane, the quantity $|1+\Gamma(z)|$ can graphically be interpreted as



As we move along the TL away from the load in the -*z* direction:

1. From (3) we see that $\left|\Gamma(z)\right| = \left|\Gamma_L e^{j2\beta z}\right| = \left|\Gamma_L\right|$, which is constant. Hence, $\Gamma(z)$ traces a circle in this plane.
2. There is CW rotation, rather than CCW, because of the factor $e^{j2\beta z}$ in (3) and our movement in the –*z* direction when moving towards the source.

Both of these facts are illustrated in the **crank diagram** shown above.

Also notice from the crank diagram that we obtain the same $\Gamma(z)$ value every $2\beta z = 2\pi$ or $\beta z = \pi$ rad movement along the TL. Consequently, from (6) we will measure the same $\left|V(z)\right|$ every $\beta z = \pi$ rad movement along any (lossless) TL. This makes sense since we're only looking at the ***magnitude*** of the voltage.

Why is $\left|V(z)\right|$ important? Because this quantity is "easy" to measure accurately. For example, using a square law detector.

Voltage Standing Wave Ratio

As we've seen repeatedly in our studies of TLs, there is generally some amount of reflection of voltage and current waves from loads attached to a TL.

To help quantify the amount of wave interference that exists on a TL, we define the voltage standing wave ratio (VSWR) as

$$\text{VSWR} \equiv \frac{\left|V(z)\right|_{\max}}{\left|V(z)\right|_{\min}} \tag{7}$$

where $\left|V(z)\right|_{\max}$ and $\left|V(z)\right|_{\min}$ are the maximum and minimum voltage magnitudes, respectively, found anywhere on a long TL.

Using (6) and the crank diagram above, we can easily determine expressions for these quantities. Specifically, we can see that

$$\left|V(z)\right|_{\max} = \left|V_o^+\right|\left(1+\left|\Gamma_L\right|\right)$$

and

$$\left|V(z)\right|_{\min} = \left|V_o^+\right|\left(1-\left|\Gamma_L\right|\right)$$

Substituting these into the definition of VSWR in (7) gives

$$\boxed{\text{VSWR}=\frac{1+\left|\Gamma_L\right|}{1-\left|\Gamma_L\right|}} \tag{8}$$

From this expression, we can definitely see that VSWR is intimately related to the amount of reflection at the load (through Γ_L) and the subsequent interference on the TL.

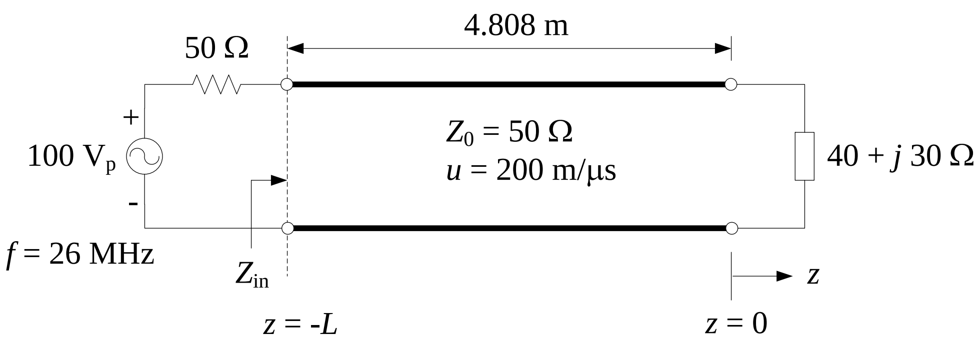
Special cases:

1. If $Z_L = 0$ (short circuit load) then $\Gamma_L = -1$. Consequently, $\left|\Gamma_L\right| = 1 \quad \Rightarrow \quad \text{VSWR} = \infty$,
2. If $Z_L = \infty$ (open circuit load) then $\Gamma_L = 1$. Consequently, $\left|\Gamma_L\right| = 1 \quad \Rightarrow \quad \text{VSWR} = \infty$,

3. If $Z_L = Z_0$ (matched load) then $\Gamma_L = 0$. Consequently, $|\Gamma_L| = 0 \;\Rightarrow\; \text{VSWR} = 1$.

Regardless of the load, $\boxed{1 \le \text{VSWR} \le \infty}$.

Example N20.1: For the TL shown below, determine the VSWR on the TL, the time averaged power delivered to the load, and the voltage at the load.



For this TL

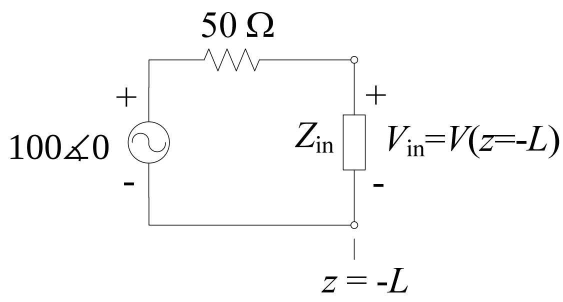
$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{40 + j30 - 50}{40 + j30 + 50} = \frac{j}{3}$$

Therefore,

$$\text{VSWR} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = \frac{1 + 1/3}{1 - 1/3} = 2$$

To determine the time averaged power delivered to the load, we'll compute the time averaged power at the TL input. Because the TL is lossless, these two quantities will be the same.

We can construct an equivalent lumped element circuit at the TL input as:



From (6) in the previous lecture

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta L)}{Z_0 + jZ_L \tan(\beta L)}$$

With

$$\beta L = \frac{\omega}{u} L = \frac{2\pi \cdot 26\times 10^6}{200\times 10^6} \cdot 4.808 = 3.927 \text{ rad} \Rightarrow \tan(\beta L) = 1.001$$

then,

$$Z_{in} = 50 \cdot \frac{40 + j30 + j50}{50 + j40 - 30} = 50 \cdot (2 + j0) = 100 \ \Omega$$

Very curious result! Z_L is complex, but Z_{in} is purely real. This is an example that TLs act as impedance transformers.

Referring to the equivalent circuit above,

$$V_{in} = \frac{Z_{in}}{Z_{in} + Z_s} \cdot V_s = \frac{100}{100 + 50} \cdot 100 = 66.67 \text{ V}$$

Since this is a lossless TL, all the time averaged power at the input to the TL will be delivered to the load. Therefore,

$$P_{AV} = \frac{1}{2}\text{Re}\left[V(-L)\cdot I(-L)^*\right] = \frac{1}{2}\text{Re}\left[V_{\text{in}}\cdot\frac{V_{\text{in}}^*}{Z_{\text{in}}^*}\right] = \frac{1}{2}\text{Re}\left[\frac{\left|V_{\text{in}}\right|^2}{Z_{\text{in}}^*}\right]$$

or

$$P_{AV} = \frac{\left|V_{\text{in}}\right|^2}{2}\text{Re}\left[\frac{1}{Z_{\text{in}}^*}\right]$$

In this example, the time averaged power delivered to the load is

$$P_{AV} = \frac{66.67^2}{2}\text{Re}\left[\frac{1}{100}\right] = 22.22 \text{ W}$$

To determine the voltage at the load, we begin with (4)

$$V(z) = V_o^+ e^{-j\beta z}\left[1+\Gamma(z)\right] = V_o^+ e^{-j\beta z}\left[1+\Gamma_L e^{j2\beta z}\right] \qquad (9)$$

The only unknown quantity in this equation is V_o^+. We can determine this complex constant by applying the boundary condition at the source location on the TL. In particular, at the input to the TL from (9)

$$V(z=-L) = \underbrace{V_{\text{in}}}_{=66.67} = V_o^+ \underbrace{e^{j\beta L}}_{1\measuredangle 3.937 \text{ rad}}\left(1+\underbrace{\Gamma_L e^{-j2\beta L}}_{=1/3}\right)$$

Solving this equation for V_o^+ we find

$$V_o^+ = \frac{66.67\left(1\measuredangle -3.937 \text{ rad}\right)}{1+1/3} = 50 \text{ V}\measuredangle 2.356 \text{ rad}$$

Substituting this result back into (9) for z = 0, we determine the voltage at the load to be

$$V_L = V(z=0) = V_o^+\left(1+\Gamma_L\right) = 50\measuredangle 2.356\cdot\left(1+\frac{j}{3}\right)$$

or

$$V_L = -47.14 + j23.58 \text{ V} = 52.70 \text{ V}\measuredangle 2.68 \text{ rad}$$
$$= 52.70 \text{ V}\measuredangle 153.6^{\circ}$$

With VisualEM

1. Confirm calculations in this example using the "Section 7.3/Problem 7.3.1" worksheet. This is a useful TL calculator.
2. See the plot of the voltage magnitude and phase. VSWR: what does it mean?
3. To better understand this TL behavior, also see the "Example 7.5" worksheet associated with Lecture 17 in these notes. Enter the revised numbers for this example:
   - Animate the voltage on the TL,
   - This **total** voltage is the sum of waves traveling in opposite directions on the TL. See the second animation in this worksheet:
     - ✓ The maximum amplitude occurs when the +z and –z waves add "in phase."
     - ✓ The minimum amplitude occurs when the +z and –z waves add "out of phase."

Section 7.3 and Problem 7.3.1

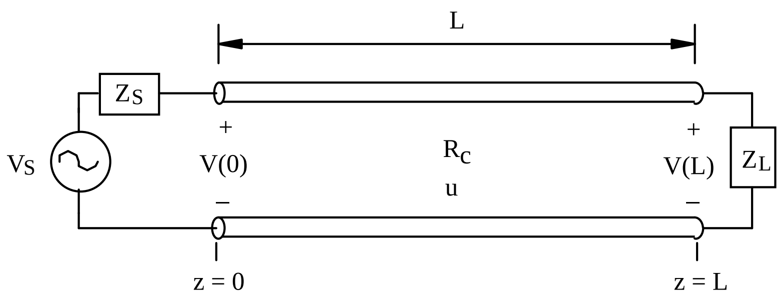
Lossless Transmission Line Calculator

Purpose

To provide a transmission-line "calculator" to compute many quantities associated with the sinusoidal excitation of lossless transmission lines. These quantities include reflection coefficients, VSWR, input impedance and time-average power delivered to a load. The magnitude and phase of the voltage on the transmission line is also plotted.

Enter parameters

The lossless transmission line is assumed to have the following arrangement for the source the load as shown:



Choose the parameters for the TL including the length, characteristic resistance and the propagation velocity:

$L := 4.808$ Length of the TL (m).

$R_C := 50$ TL characteristic resistance (Ω).

$u := 200 \cdot 10^6$ Propagation velocity (m/s).

Also choose the parameters for the source and the load:

$V_S := 100 \cdot \exp(j \cdot 0 \cdot deg)$ Source open-circuit voltage (V, rad).

$f := 26 \cdot 10^6$ Source frequency (Hz).

$Z_S := 50 + j \cdot 0$ Source impedance (Ω).

$Z_L := 40 + j \cdot 30$ Load impedance (Ω).

Compute the phase constant, β, of the voltage and current waves on this TL:

$$\omega := 2\cdot\pi\cdot f$$

$$\beta := \frac{\omega}{u} \qquad \beta = 0.817 \qquad \text{(rad/m)}$$

Lossless TL calculator

As requested in Prob. 7.3.1, there are a number of quantities that are to be computed for thi transmission line. These are computed below under the corresponding headings as given in homework problem description.

- *Line length as a fraction of a wavelength*

The wavelength on this TL is:

$$\lambda := \frac{u}{f} \qquad \lambda = 7.692 \qquad \text{(m)}$$

Therefore the line length in units of wavelengths is:

$$\frac{L}{\lambda} = 0.625$$

- *Voltage reflection coefficient at the load and at the TL input*

Using Equation (66) in Chap. 7 of the text, the voltage reflection coefficient at the load

$$\Gamma_L := \frac{Z_L - R_C}{Z_L + R_C} \qquad \Gamma_L = 0.333j$$

The (generalized) voltage reflection coefficient is given in (68) of Chap. 7:

$$\Gamma(z) := \Gamma_L\cdot\exp\left[j\cdot 2\cdot\beta\cdot(z - L)\right]$$

whereby we can compute the reflection coefficient at the input to the TL as:

$$\Gamma(0) = 0.333 - 1.676j \times 10^{-4}$$

- *Voltage Standing Wave Ratio (VSWR)*

From Equation (105) in Chap. 7 of the text:

$$\text{VSWR} := \text{if}\left(\left|\Gamma_L\right| \neq 1, \frac{1 + \left|\Gamma_L\right|}{1 - \left|\Gamma_L\right|}, \infty\right) \qquad \text{VSWR} = 2.00$$

- *The input impedance to the line*

From Equation (62) in Chap. 7 of the text:

$$Z_{in} := R_C \cdot \frac{1 + \Gamma(0)}{1 - \Gamma(0)} \qquad Z_{in} = 100.000 - 0.038j \qquad (\Omega)$$

- *Time-domain voltage at the line input and at the load*

Using Equation (72a) in Chap. 7 of the text:

$$\Gamma_S := \frac{Z_S - R_C}{Z_S + R_C}$$ Source-end reflection coefficient.

$$V(z) := \frac{1 + \Gamma_L \cdot \exp[-j \cdot 2 \cdot \beta \cdot (L - z)]}{1 - \Gamma_S \cdot \Gamma_L \cdot \exp(-j \cdot 2 \cdot \beta \cdot L)} \cdot \frac{R_C}{Z_S + R_C} \cdot V_S \cdot \exp(-j \cdot \beta \cdot z)$$

Using V(z), then at the input to the TL:

$$|V(0)| = 66.667$$ Voltage magnitude, TL input (V).

$$\text{if}\left(|V(0)| \neq 0, \frac{\arg(V(0))}{\deg}, 0\right) = -7.200 \times 10^{-3}$$ Voltage phase, TL input (°).

and at the load:

$$|V(L)| = 52.705$$ Voltage magnitude, TL load (V).

$$\text{if}\left(|V(L)| \neq 0, \frac{\arg(V(L))}{\deg}, 0\right) = 153.421$$ Voltage phase, TL load (°).

- *Time-average power delivered to the load*

At the load, the ratio of the total voltage and current is equal to the load impedance. We have an expression for the voltage anywhere on the TL already defined in the previous subheading. Therefore, the current at the load is:

$$I_L := \frac{V(L)}{Z_L}$$

The time-average power in the +z direction is given in Equation (107) of Chap. 7 as:

$$P_{av}(z) := \frac{1}{2} \cdot \text{Re}\left(V(L) \cdot \overline{I_L}\right)$$

The time-average power delivered to the load is then:

$$P_{av}(L) = 22.22222 \qquad (W)$$

Plot phasor-domain voltage and current on the TL

We will now plot the magnitude and phase of the voltage everywhere on the TL. Choose th number of points at which to plot the voltage:

$npts := 300$ Number of points to plot in z.

$z_{start} := 0 \qquad z_{end} := L$ z starting and ending points (m).

Construct a list of z_i values at which to plot the voltage:

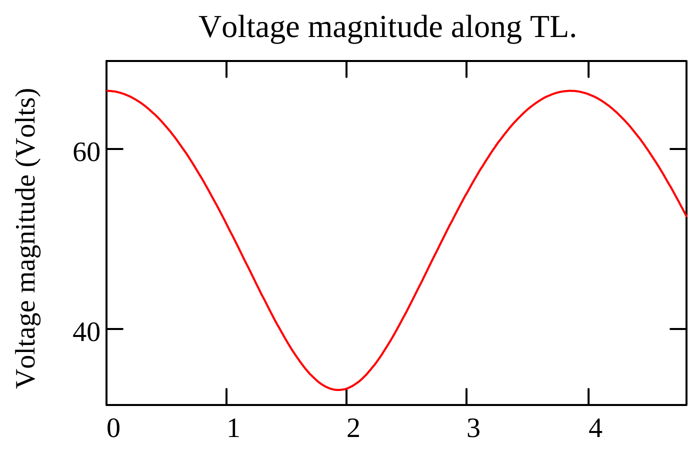
$$i := 0 .. npts - 1 \qquad z_i := z_{start} + i \cdot \frac{z_{end} - z_{start}}{npts - 1}$$

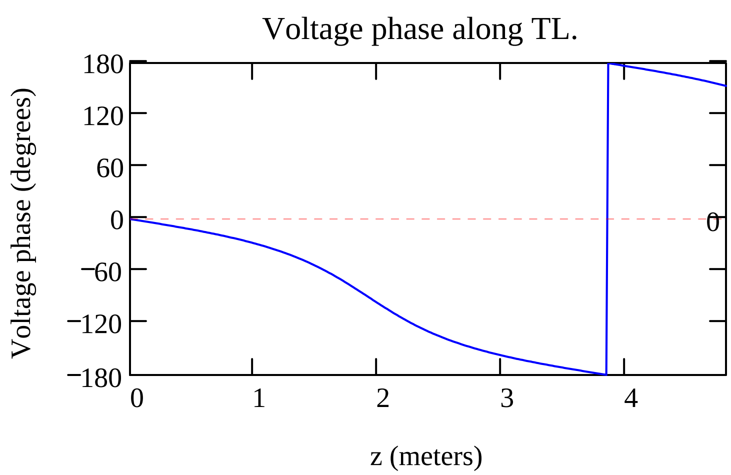
Compute the voltage magnitude and phase at every position z_i along the TL:

$Mag_{V_i} := \left|V(z_i)\right|$ Voltage magnitude.

$\theta_{V_i} := \text{if}\left(\left|V(z_i)\right| \neq 0, \frac{\arg(V(z_i))}{deg}, 0\right)$ Voltage phase (°).

Now plot the magnitude and phase of the voltage along the TL:





This voltage exists on a TL with $L = 4.808$ (m), $R_C = 50$ (Ω) and $Z_L = 40.0 + 30.0j$ (Ω) where the voltage source is located at the left-hand edge of these plots and the load at the right-hand edge.

The values of the voltage at the source and load ends of the TL can be confirmed by measuring the magnitude and phase in these two plots and comparing these values with those computed earlier in this worksheet.

End of worksheet.

