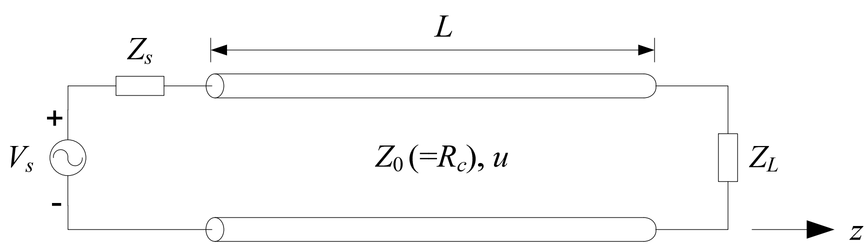
Lecture 17: Sinusoidal Steady State Excitation of Lossless Transmission Lines.

We will continue our TL studies by considering the steady state response of TLs to sinusoidal excitation. Communication systems and power lines are two examples of industries that use sinusoids or modified sinusoids.

Consider the following TL in the sinusoidal steady state:



We previously derived in Lecture 11 the wave equations for the voltage and current as

$$\frac{\partial^2 V(z,t)}{\partial z^2} = \frac{1}{u^2}\frac{\partial^2 V(z,t)}{\partial t^2} \tag{1}$$

and

$$\frac{\partial^2 I(z,t)}{\partial z^2} = \frac{1}{u^2}\frac{\partial^2 I(z,t)}{\partial t^2} \tag{2}$$

For the sinusoidal steady state, we will employ the phasor representation of the voltage and current as

$$V(z,t) = \text{Re}\left[V(z)e^{j\omega t}\right] \tag{3}$$

$$I(z,t) = \text{Re}\left[I(z)e^{j\omega t}\right] \tag{4}$$

where $V(z)$ and $I(z)$ are phasors. Notice that these phasors are functions of position along the TL.

Substituting (3) into (1) gives

$$\frac{d^2V(z)}{dz^2} = \frac{1}{u^2}(j\omega)^2 V(z) = -\frac{\omega^2}{u^2}V(z) \tag{5}$$

We define

$$\boxed{\beta = \omega\sqrt{lc} \quad \text{[rad/m]}} \tag{6}$$

as the phase constant for reasons that will be apparent shortly. (In this equation, l and c are the typical per-unit-length parameters of the TL.) From (6)

$$\beta^2 = \omega^2 lc = \frac{\omega^2}{u^2}$$

Substituting this last result into (5) gives

$$\frac{d^2V(z)}{dz^2} + \beta^2 V(z) = 0 \tag{7}$$

Similarly, from (4) and (2) we can derive

$$\frac{d^2I(z)}{dz^2} + \beta^2 I(z) = 0 \tag{8}$$

Equations (7) and (8) are the wave equations for V and I in the frequency domain (i.e., the phasor domain).

The solutions to these two second order ordinary differential equations are

$$V(z) = V^+ e^{-j\beta z} + V^- e^{+j\beta z} \tag{9}$$

and
$$I(z) = I^+ e^{-j\beta z} + I^- e^{+j\beta z} \tag{10}$$
where V^+, V^-, I^+, and I^- are complex constants.

Confirm Solution

We can confirm the correctness of these solutions by direct substitution into (7) and (8). For example, substituting $V^+ e^{-j\beta z}$ from (9) into (7) gives
$$V^+(-j\beta)^2 e^{-j\beta z} + \beta^2 V(z) \stackrel{?}{=} 0$$
or
$$-\beta^2 V^+ e^{-j\beta z} + \beta^2 V^+ e^{-j\beta z} \stackrel{?}{=} 0$$
which is indeed true. Therefore, $V^+ e^{-j\beta z}$ in (9) is a valid solution to (7).

Current Wave Amplitudes

The constants I^+ and I^- in (10) can be expressed in terms of V^+ and V^-. (We saw similar behavior in the time domain analysis of TLs in Lecture 12.) In particular, it can be shown that
$$I^+ = \frac{V^+}{Z_0} \tag{11}$$
and
$$I^- = -\frac{V^-}{Z_0} \tag{12}$$

Here we are using Z_0 to indicate the characteristic impedance of the TL. For so-called lossless TLs

$$Z_0 = R_c = \sqrt{\frac{l}{c}} \quad [\Omega] \tag{13}$$

It is customary to use the symbol Z_0 for sinusoidal steady state TL problems rather than R_c.

If we substitute (11) and (12) into (10) we find that

$$V(z) = V^+ e^{-j\beta z} + V^- e^{+j\beta z} \tag{14}$$

and

$$I(z) = \frac{1}{Z_0} V^+ e^{-j\beta z} - \frac{1}{Z_0} V^- e^{+j\beta z} \tag{15}$$

Both of these equations should be committed to memory.

The first terms in (14) and (15) are the phasor representation of waves propagating in the +z direction along the TL. The second terms in both equations represent waves propagating in the –z direction.

Discussion

- As stated above, the first terms in (14) and (15) are the phasor representation of waves traveling in the +z direction. To see this, convert the first term in (14) to the time domain:

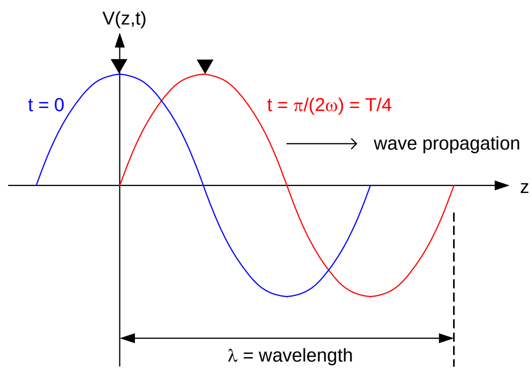
$$V(z,t) = \text{Re}\left[V^+ e^{-j\beta z} e^{j\omega t}\right] = \text{Re}\left[\left|V^+\right| e^{j\theta^+} e^{j(\omega t - \beta z)}\right]$$
$$= \left|V^+\right| \cos\left(\omega t - \beta z + \theta^+\right)$$
$$= \left|V^+\right| \cos\left[\omega\left(t - \frac{\beta}{\omega} z\right) + \theta^+\right]$$
$$= \left|V^+\right| \cos\left[\omega\left(t - \frac{z}{u}\right) + \theta^+\right]$$

We can clearly see in this last result that we have a function of time with argument $t - z/u$. From our previous discussions with TLs (see Lecture 12) we recognize that this is a wave propagating in the +z direction with speed u.

- Similarly, we can show that $V^- e^{+j\beta z}$ and $I^- e^{+j\beta z}$ are phasor representations of waves propagating in the –z direction.

Phase Constant and Wavelength

The sinusoidal steady state voltage on a TL is shown here drawn at two different times t:



Let us fix our attention to one "point" on the wave (▼, for example). This requires that

$$\cos(\omega t - \beta z) = \text{constant} \tag{16}$$

From (16) we can deduce a "dual" relationship between ωt and βz. Think of it this way: since both terms ωt and βz must have the same "units" (radians), then β is the spatial analog to ω. That is, the phase constant β indicates how many radians the argument will change per unit of distance (rather than per unit of time, as for ω).

The period of the spatial waveform is called the wavelength λ equal to

$$\boxed{\lambda = \frac{2\pi}{\beta} \text{ [m]}} \tag{17}$$

Another way of viewing the wavelength λ is it's the distance a constant phase front ($\phi \equiv \text{phase} = \omega t - \beta z = \text{constant}$) of the wave travels in one complete time period T.

For example, at

$$t = t_1: \quad \phi_1 = \omega t_1 - \beta z_1$$

while at

$$t = t_2 = t_1 + T: \quad \phi_2 = \omega(t_1 + T) - \beta z_2$$

Because $\phi_1 = \phi_2$ (i.e., we're observing the same relative position on the wave at two different times), then

$$\omega t_1 - \beta z_1 = \omega(t_1 + T) - \beta z_2$$

or
$$\omega T = \beta\left(z_2 - z_1\right)$$

It is known that $\omega T = 2\pi$, and we'll define this distance $z_2 - z_1$ as the wavelength. Hence,
$$\lambda = \frac{2\pi}{\beta}$$
as in (17).