Lecture 13: Transmission Line Termination, Reflections. Current Waves.

The TLs we've considered so far were semi-infinitely long. This is useful when trying to understand basic behavior such as *V* and *I* wave propagation, for example. Obviously semi-infinite TLs are not very practical.

As an example of a practical application, interconnected ICs on a PCB provide "termination" of TLs.

We will see in this lecture that terminations of a TL generally will cause the voltage and current waves to (partially) reflect from the load.

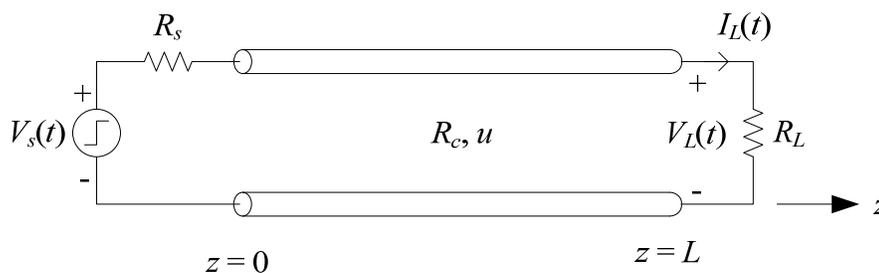Consider a TL with resistive termination and the same step input voltage from the last example:

Fig. 1

As we saw in the last example, a step voltage is launched at time $t = 0$ with amplitude $V_0 \cdot R_c / (R_c + R_s)$. Here, the disturbance will propagate to the right along the TL until it encounters R_L.

When the disturbance reaches the load, there will be created a new voltage disturbance that propagates to the left (-z direction). This is called the reflected wave.

Voltage Reflection Coefficient

What is the amplitude of the reflected voltage wave? To answer this very important question we begin with the general TL solutions from Lecture 12:

$$V(z,t) = V^+\left(t - \tfrac{z}{u}\right) + V^-\left(t + \tfrac{z}{u}\right) \tag{1}$$

$$I(z,t) = \frac{1}{R_c}V^+\left(t - \tfrac{z}{u}\right) - \frac{1}{R_c}V^-\left(t + \tfrac{z}{u}\right) \tag{2}$$

At the load $z = L$ these two equations evaluate to:

$$V(z = L,t) = V^+\left(t - \tfrac{z^L}{u}\right) + V^-\left(t + \tfrac{z^L}{u}\right) \tag{3}$$

$$I(z = L,t) = \frac{1}{R_c}V^+\left(t - \tfrac{z^L}{u}\right) - \frac{1}{R_c}V^-\left(t + \tfrac{z^L}{u}\right) \tag{4}$$

However, at the load we also have the resistor's lumped element terminal relationship from Ohm's law:

$$V_L(t) = R_L I_L(t) \tag{5}$$

We can see from the figure above that

$$V_L(t) = V(z = L,t) \quad \text{in (3)}$$
$$I_L(t) = I(z = L,t) \quad \text{in (4)}$$

Note one subtle point here. Both RHSs above are the sum of $+z$ and $-z$ propagating waves! At $z = L$, both of these waves **sum** to give the terminal voltage V_L and the terminal current I_L.

Now, substituting these two equations (3) and (4) into (5) gives

$$V^+\left(t - \tfrac{L}{u}\right) + V^-\left(t + \tfrac{L}{u}\right) = \frac{R_L}{R_c}V^+\left(t - \tfrac{L}{u}\right) - \frac{R_L}{R_c}V^-\left(t + \tfrac{L}{u}\right)$$

Dividing by $V^+\left(t + L/u\right)$ gives

$$1 + \frac{V^-\left(t + \tfrac{L}{u}\right)}{V^+\left(t - \tfrac{L}{u}\right)} = \frac{R_L}{R_c}\left[1 - \frac{V^-\left(t + \tfrac{L}{u}\right)}{V^+\left(t - \tfrac{L}{u}\right)}\right]$$

Solving this equation for the ratio V^- / V^+ results in

$$\frac{V^-\left(t + \tfrac{L}{u}\right)}{V^+\left(t - \tfrac{L}{u}\right)} = \frac{R_L - R_c}{R_L + R_c} \tag{6}$$

This ratio is incredibly important to our work with TLs. Consequently, we will define this ratio by the symbol Γ_L as

$$\Gamma_L \equiv \frac{V^-\left(t + \tfrac{L}{u}\right)}{V^+\left(t - \tfrac{L}{u}\right)} \tag{7}$$

which, we can deduce, is the **voltage reflection coefficient** at the load position $z = L$. In terms of Γ_L, (6) becomes

$$\boxed{\Gamma_L = \frac{R_L - R_c}{R_L + R_c}} \tag{8}$$

Discussion

- Some general characteristics of the voltage reflection coefficient are:
  - Special cases for Γ_L:

    1. If $R_L = \infty$ (open circuit) $\Rightarrow$ $\Gamma_L = 1$ $\left(= \frac{V^-}{V^+}\Big|_{z=L} \right)$

    2. If $R_L = 0$ (short circuit) $\Rightarrow$ $\Gamma_L = -1$ $\left(= \frac{V^-}{V^+}\Big|_{z=L} \right)$

    3. If $R_L = R_c$ ("matched load") $\Rightarrow$ $\Gamma_L = 0$ $\left(= \frac{V^-}{V^+}\Big|_{z=L} \right)$

    In this last case, there is no reflection of the wave from the load!
  - $-1 \leq \Gamma_L \leq 1$

- Note from (8) that the voltage reflection coefficient is only a function of the load resistance and the characteristic resistance of the TL. Γ_L is not dependent on the amplitude of the incident voltage. Therefore, whenever a voltage V^+ is incident on the load, a reflected voltage V^- equal to

$$V^- \left(t + \tfrac{L}{u} \right) = \Gamma_L V^+ \left(t - \tfrac{L}{u} \right)$$

  is launched that propagates in the -z direction.

- This reflected voltage adds to the incident voltage to produce the total voltage anywhere on the TL.

- When this reflected voltage disturbance reaches the source, there will be another reflection of the voltage wave

$$\boxed{\Gamma_s \equiv \frac{V^+\left(t - \frac{z^0}{u}\right)}{V^-\left(t + \frac{z^0}{u}\right)} = \frac{R_s - R_c}{R_s + R_c}} \tag{9}$$

where R_s is the source resistance.

- This process of the voltage wave reflecting off the source, propagating down the TL, then reflecting from the load, then propagating down the TL, then reflecting from the source, etc. repeats indefinitely.

Example N13.1. A previously discharged TL shown below is excited by a DC source of 30 V at time $t = 0$. Sketch the voltage along the TL at times 1 μs, 2.5 μs, 4.5 μs, and 6.5 μs.
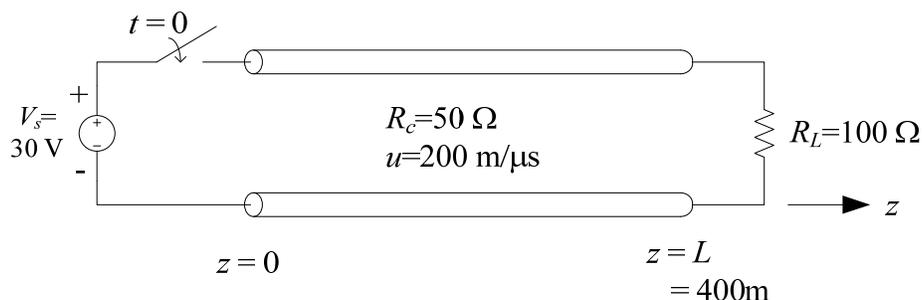
The voltage reflection coefficients are:
- At the load, from (8):

$$\Gamma_L = \frac{R_L - R_c}{R_L + R_c} = \frac{100 - 50}{100 + 50} = \frac{50}{150} = \frac{1}{3}$$

- At the source, from (9):

$$\Gamma_s = \frac{R_s - R_c}{R_s + R_c} = \frac{0 - 50}{0 + 50} = -1$$

The time to transit the TL is

$$t_L = \frac{L}{u} = \frac{400 \text{ m}}{200 \text{ m}/\mu s} = 2 \ \mu s$$

See the sketches on the next page for the solution:

- ② The wave reflects off the load with $\Gamma_L = 1/3$. The wave front then propagates to the left. (Can think of this as originating at $z = 2L$, if you wish.)
- The total voltage equals the sum of the incident and all reflected waves.
- ③ The wave front reflects off the source. Complete reflection with $\Gamma_L = -1$.
- ④ The wave front propagates down the TL to the right and then reflects again off the load ($\Gamma_L = 1/3$).

We might call this approach the "time evolution method" to solving for the voltage on the TL with a unit step input voltage.

See *VisualEM* "Example 7.2 Worksheet" [set n$_{skip}$=1, W=20 μs (=10t_L)]:

- Animation in time.
- Compare solution on next page at $t = 1$ μs and $t = 2.5$ μs.
- Voltage is converging to $V(z,t) \approx 30$ V. Is this expected?

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$\bullet \; \underline{\Gamma_L} = \dfrac{R_L - R_c}{R_L + R_c} = \dfrac{100 - 50}{100 + 50} = \dfrac{1}{3}$

$\bullet \; \underline{\Gamma_S} = \dfrac{R_S - R_c}{R_S + R_c} = \dfrac{0 - 50}{0 + 50} = -1$

$\bullet \; \underline{t_L} = \dfrac{L}{u} = \dfrac{400 \, m}{200 \frac{m}{\mu s}} = \underline{2 \, \mu s}$

first reflection from load has yet to reach the TL.
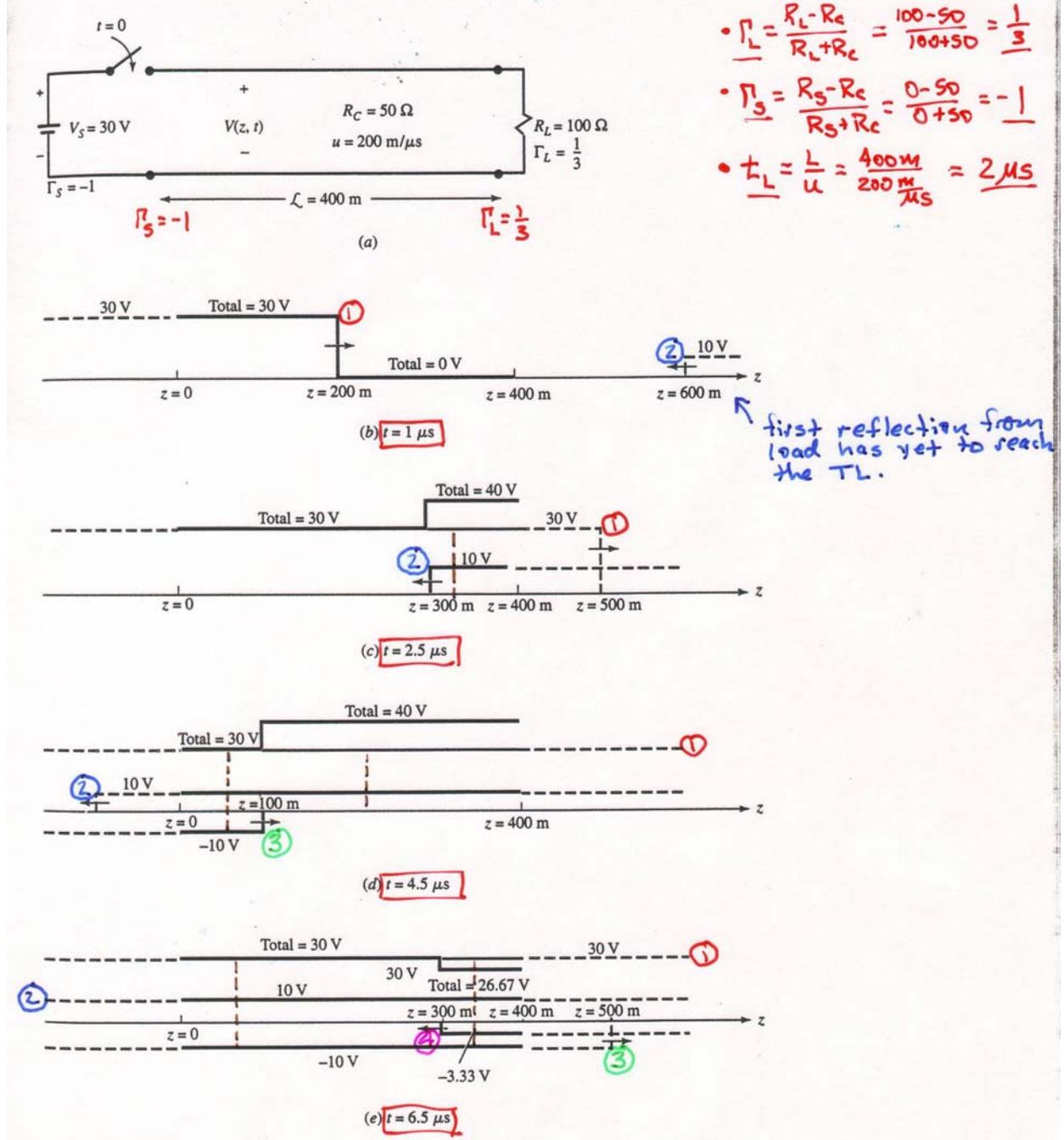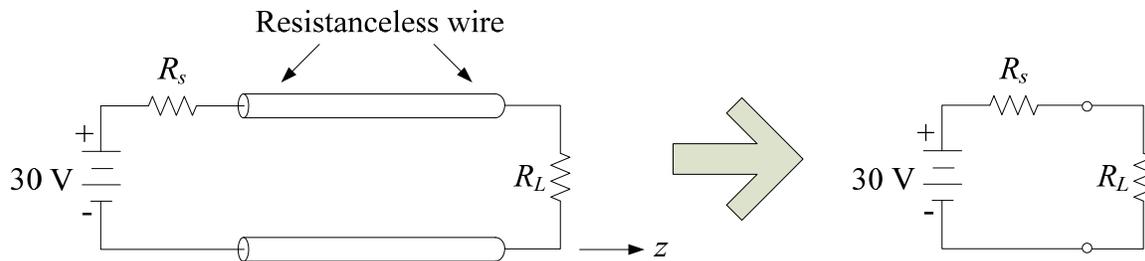
FIGURE 7.12
Example 7.1.

Discussion

- $V(z,t) \approx 30$ V everywhere on the TL for $t \gg t_L$. This is the "steady state." Do we expect a nonzero steady state voltage? Yes, since the source remains 30 V for $t > 0$. For $t \gg t_L$, have

  In steady state, this is just a simple resistive circuit with $R_s = 0$, maybe two parallel wires or a coaxial cable, for example, interconnecting a battery and a resistor.
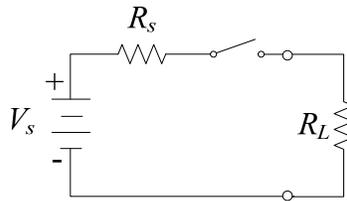
- In steady state on the TL, notice that
$$\frac{V(z,t)}{I(z,t)} \neq R_c$$
  Why is this? Because
$$R_c = \frac{V^+(z,t)}{I^+(z,t)} \neq \frac{V(z,t)}{I(z,t)}$$

- Now that you've observed the fundamental nature of electricity as waves, you can begin to appreciate the complexity of the transient results shown above. In a beginning electrical circuits class, such a resistive circuit as

would not have a "transient" response since there is no energy storage component (C or L) to charge up.

But, in fact, there is a transient response much more basic than that acting in a real-life circuit, as we have just observed in this example. When the switch is closed in the circuit above, an electromagnetic wave is launched that propagates back and forth in the interconnecting wires, traveling at the speed of light in that structure, that quickly reaches steady state in nano- or micro-seconds.

But the final, steady state response is a simple DC one you could easily have calculated beforehand – but the transient results leading up to this is very complex. Fascinating!

Current Waves

So far we've only considered the propagation and reflection of voltage waves on a TL. We know, however, that with a voltage wave there must also be an associated current wave: Because of

the laws of electromagnetics, we cannot have one without the other.

Consider once again the TL of Fig. 1 excited by a unit-step voltage source. The current on the TL is expressed by (2). When the source initially turns on at $t = 0$, we can construct an equivalent circuit at the input to the TL in order to determine the amplitude of the initial current wave amplitude I^+. In the case of $V_s(t) = V_0 u(t)$ the equivalent circuit is
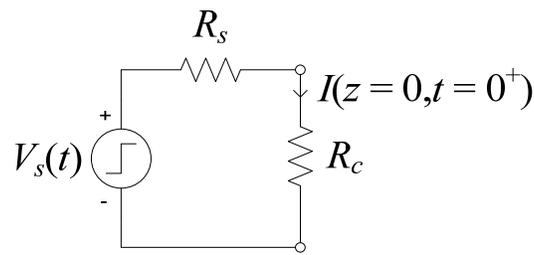
From this circuit, we can determine that

$$I\left(z = 0, t = 0^+\right) = \frac{V_0}{R_s + R_c} = I^+ \tag{10}$$

A current wave with this amplitude I^+ in (10) then propagates down the TL in Fig. 1 until it encounters the resistive load. Like the voltage wave, this current wave is partially reflected by the load and the remaining energy is "transmitted" into the load resistance R_L.

We can determine this current reflection amplitude by beginning with (6)

$$\frac{V^-\left(t+\frac{L}{u}\right)}{V^+\left(t-\frac{L}{u}\right)} \equiv \Gamma_L = \frac{R_L - R_c}{R_L + R_c} \tag{11}$$

As we learned in Lecture 12, the amplitudes of V^+ and V^- are related to I^+ and I^- as

$$I^+\left(t-\tfrac{L}{u}\right) = \frac{V^+\left(t-\frac{L}{u}\right)}{R_c} \tag{12}$$

and

$$I^-\left(t+\tfrac{L}{u}\right) = -\frac{V^-\left(t+\frac{L}{u}\right)}{R_c} \tag{13}$$

Substituting these relationships into (11), we find that

$$\boxed{\frac{I^-\left(t+\frac{L}{u}\right)}{I^+\left(t-\frac{L}{u}\right)} = -\frac{R_L - R_c}{R_L + R_c} = -\Gamma_L} \tag{14}$$

This result in (14) tells us that the current reflection coefficient at the load is the negative of the load voltage reflection coefficient. It turns out that this is a general result: Current reflection coefficients are the negative of the corresponding voltage reflection coefficients.
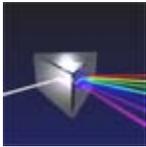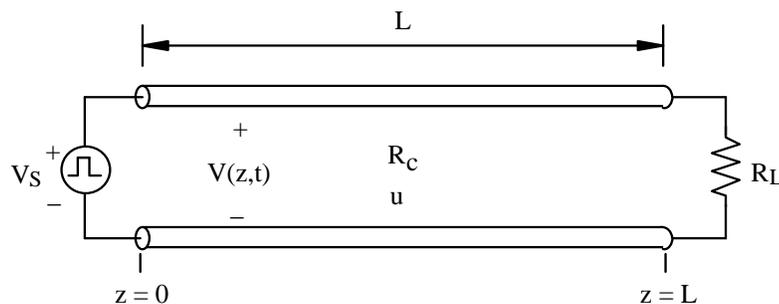
Example 7.2

Numerical Solution for the Propagation of a Voltage Pulse on a Transmission Line

Purpose

To compute and visualize the propagation of a voltage pulse on a transmission line (TL). Two types of plots are generated in this worksheet. The first is an animation of the voltage on the TL versus time. The second is a plot of the voltage at a specific location on the TL versus time. The computation of the voltage on the TL is accomplished using a numerical solution called the finite difference time domain method. (Requires Mathcad 8 Professional, Academic or Explorer.)

Background

A pictorial representation of the lossless transmission line (TL) is shown in Fig. 7.12a in the text and in the figure below:

As given in Example 7.2, the source generates a voltage pulse V_S at z = 0 that changes from zero volts to some value for a specified length of time and then switches back to zero. When the source voltage transitions to the "on" state at time t = 0, a voltage propagates down the TL as discussed in Section 7.2 of the text. By accounting for the propagation speed of this voltage disturbance and the reflection from the load and source ends of the TL, it is possible to compute the voltage anywhere on the TL at any time. This is the approach used in Example 7.2.

In this worksheet, we will solve for the voltage on the TL in a different fashion. Here we will use a numerical method to compute the voltage at a discrete set of points along the TL and at a discrete set of times. This will be accomplished using the finite difference time domain (FDTD) method. A complete discussion of the FDTD method is beyond the scope of this worksheet. However, a few details will be presented so that you can effectively execute this worksheet and observe the voltage propagation along this TL and at the terminals of the TL.

In the FDTD method, the TL is divided into nz segments as shown in the figure below:
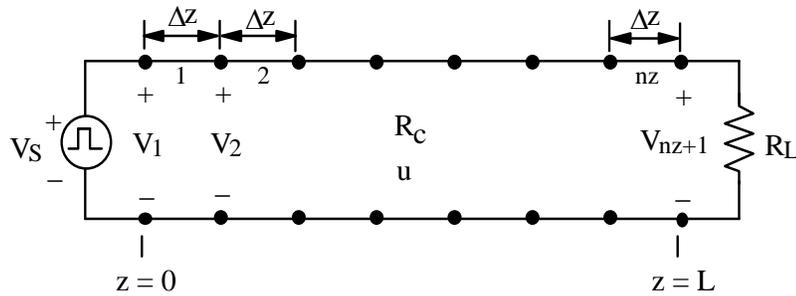
The voltage will be computed at the nz+1 "nodes" and at a discrete number of nt time instances. Each of these times is separated by an equal amount dt that is computed later in this worksheet. The FDTD solution is accomplished by applying the central difference approximation for the derivatives in space and time that appear in the governing equations for the voltage and the current in (9a) and (9b) in Chap. 7 of the text:

$$\frac{\partial V(z,t)}{\partial z} = -l\frac{\partial I(z,t)}{\partial t}$$

$$\frac{\partial I(z,t)}{\partial z} = -c\frac{\partial V(z,t)}{\partial t}$$

After enforcing the boundary conditions at the source and load ends of the TL, it is possible to obtain solutions for the voltage and current at each node of the TL at one time that are dependent only upon values of the voltage and current at previous times in the simulation. That is, this FDTD procedure requires that the voltage and current be solved for at each successive increment of time beginning, in this worksheet, at t = 0.

Enter the transmission line, voltage source and FDTD parameters

Choose the parameters for the transmission line (TL):

$L := 400$ Length of TL (m).

$u := 200 \cdot 10^6$ Propagation velocity of the TL (m/s).

$R_C := 50$ Characteristic resistance of the TL (Ω).

$R_L := 100$ Load resistance (Ω).

$R_S := 0$ Source resistance (Ω).

Using Equations (26) and (31) in Chap. 7 of the text, the per-unit-length parameters for this TL are found to be:

$$c_{tl} := \frac{1}{u \cdot R_C} \qquad c_{tl} = 1.000 \times 10^{-10} \qquad \text{Per-unit-length capacitance of TL (F/m).}$$

$$l_{tl} := c_{tl} \cdot R_C{}^2 \qquad l_{tl} = 2.500 \times 10^{-7} \qquad \text{Per-unit-length inductance of TL (H/m).}$$

Now choose the source voltage amplitude and pulse width:

$V_0 := 30$ Source voltage amplitude (V).

$W := 20 \cdot 10^{-6}$ Width (in time) of the source voltage pulse (s).

For the numerical FDTD solution for the voltage on this transmission line, choose the number of segments in which to subdivide the TL, nz, and the number of time steps to compute the solution, nt:

$nz := 40$ Number of segments to subdivide the TL.

$nt := 300$ Number of time steps to compute the solution.

Compute uniform spacing of the voltage nodes, Δz, and the time increment Δt:

$$\Delta z := \frac{L}{nz} \qquad \Delta t := \frac{\Delta z}{u}$$

The source voltage pulse is assumed to turn on at time $t = 0$. For $nt = 300$ time steps, the final time for which a voltage solution will be computed is:

$$t_{end} := nt \cdot \Delta t \qquad t_{end} = 1.500 \times 10^{-5} \qquad \text{(s)}$$

Solve for the voltage on the TL

Using these parameters for the TL, the voltage source and the number of segments (nz) and the number of time steps (nt) for the FDTD solution, we are now in position to solve for the voltage at each node on the TL at each time step. This FDTD solution is performed in the following *referenced worksheet*.

Reference:D:\Classes\EE 382 - Applied Electromagnetics\Lecture notes, EE 382\EE 382, Lecture notes, Mathcad wc

The final result from this FDTD solution is the list $V_{k,n}$ that contains the values of the voltage at the kth position in z along the TL (beginning at $z = 0$) at the nth time instant (beginning at t $= 0$).

Animated plot of V propagating on the TL

With the voltage determined at all of the nodes on the TL at the discrete list of times, we are now able to generate a number of plots to visualize these results. The first plot we will generate is an animation clip of the voltage everywhere on the TL and as a function of time.

The voltage was computed at $nt = \blacksquare$ time steps. Due to the time it takes to construct the animation of this voltage disturbance on the TL, you may not wish to have Mathcad construct the animation clip using all of these time steps. Therefore, you can set the parameter n_{skip} to specify how many time steps to disregard between successive frames when generating the animation clip. For example, if $n_{skip} = 0$, all time steps are used; if $n_{skip} = 1$, every second time step is used; if $n_{skip} = 2$, every third time is used, etc.
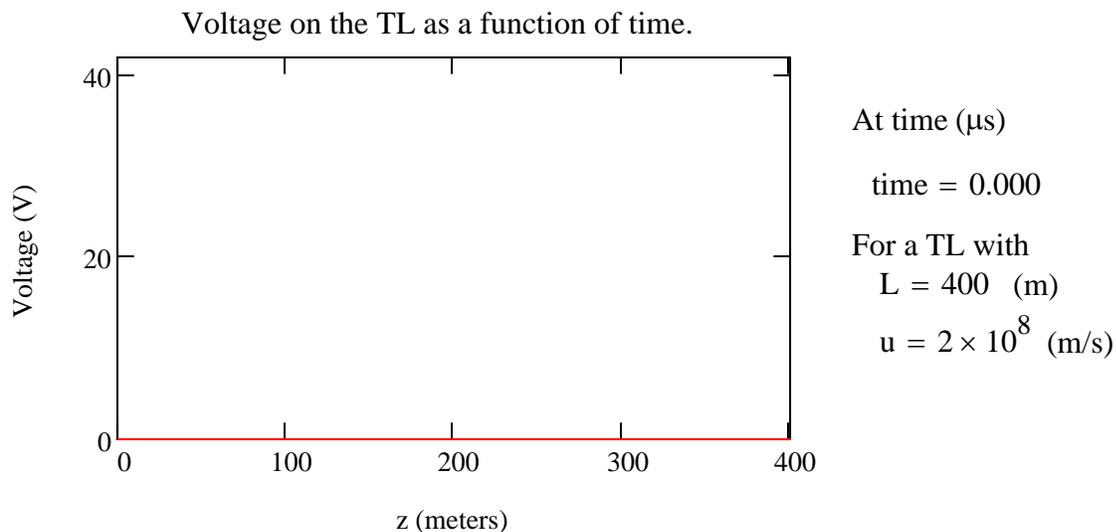
$n_{skip} := 1$ Choose the number of time steps to skip between successive frames in the animation clip.

Compute the z position of all voltage nodes in the FDTD solution and define the parameter time in terms of FRAME:

$$k := 1 .. nz + 1 \qquad z_k := (k - 1) \cdot \frac{L}{nz} \qquad time := \frac{FRAME \cdot (n_{skip} + 1) \cdot \Delta t}{10^{-6}}$$

Now generate an animation clip of the voltage on this TL. For best results, in the "Animate" dialog box choose To = 150 .

Voltage on the TL as a function of time.

At time (μs)

time = 0.000

For a TL with
  L = 400   (m)
  $u = 2 \times 10^8$ (m/s)

The source is located at the left-hand edge of this plot and the load is located at the right-hand edge. After observing this animation clip, it becomes quite clear that, indeed, this voltage disturbance does propagate down the TL! The one-way transit time for this pulse is:

$L \cdot u^{-1} = 2.000 \times 10^{-6}$ Time for voltage disturbance to traverse the TL (s).

You can also see in the animation clip that this much time is required for the leading edge of the pulse to propagate along the TL from the source to the load. This fact is most easily established by pausing the animation playback and then dragging the slider bar with the mouse until the leading edge reaches the right-hand edge of the plot. Note that depending on the number of time steps, nt, chosen earlier you may not be able to view the voltage at the

exact time instance of the one-way transit time.

Using the parameters for the TL as specified in Example 7.2, you will see that this voltage pulse propagates back and forth along the TL, but the amplitude is reduced each time this pulse reflects from the load. The reduction in the amplitude is due to some power dissipation in the nonzero and finite load resistance.

You may wish to experiment with other types of loads and observe the effects on this voltage pulse. Three interesting examples include a short circuit ($R_L = 0\ \Omega$), and open circuit ($R_L = 10^6\ \Omega$, or other very large value) and a matched load ($R_L = R_C$).

Plot V at a fixed z versus time

The final plot we will generate is the voltage everywhere on the TL but at a fixed time.

As mentioned earlier in this worksheet, the FDTD solution for the voltage was performed at a discrete set of time values beginning at $t = 0$ and each separated by

$$\Delta t = 5.000 \times 10^{-8}\quad (s)$$

We can only plot the voltage along the TL at one or more of these discrete times. This is entirely equivalent to choosing one or more of the time steps that range from 0 to $nt = 300$. At $t = 0$, the TL was assumed to be "relaxed" such that the voltage and current were zero everywhere on the TL.
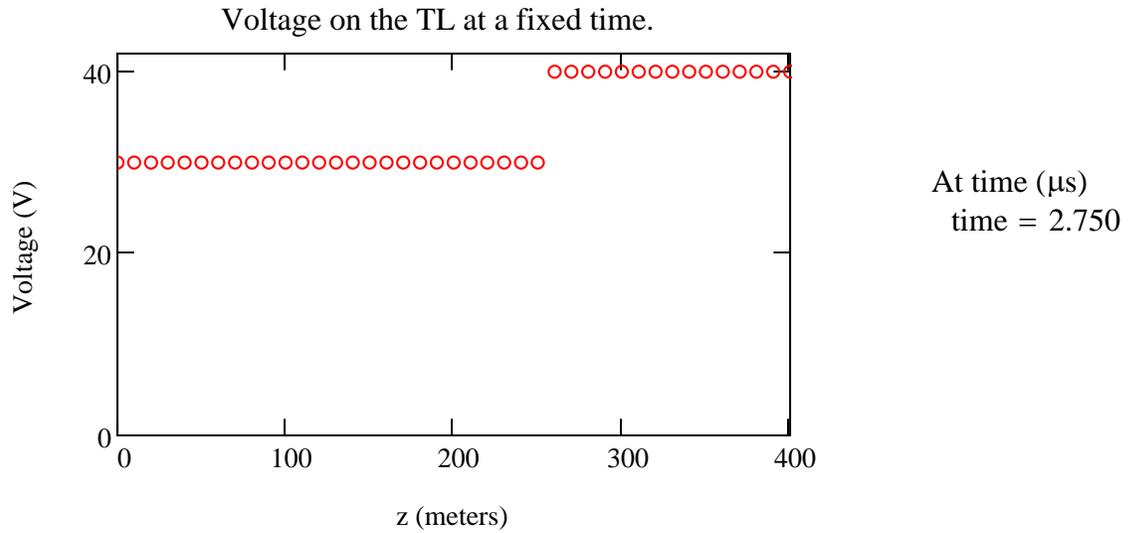
Choose one of the time steps at which to plot the voltage along the TL:

$n_{t\_plot} := 55$ 	Choose a time step between 0 and $nt = 300$.

At this particular time step, the time is

$$time := \frac{n_{t\_plot} \cdot \Delta t}{10^{-6}}\qquad time = 2.750 \qquad (\mu s)$$

Now plot the voltage everywhere on the TL at this time:

Voltage on the TL at a fixed time.

At time (μs)
time = 2.750

The length of the TL is L = 400 (m) with a propagation velocity $u = 2 \times 10^8$ (m/s).

You can compare this plot with that given in Fig. 7.13 of the text. The voltage at any of these points can be measured directly from the plot.

End of worksheet.