Lecture 12: Time Domain Solutions to TL Wave Equations.

The second-order PDEs in (9) and (10) of the last lecture are complicated. These equations are called wave equations since, as we will see in this lecture, their solutions are waves.

Voltage Wave Equation Solutions

Consider again the voltage equation

$$\frac{\partial^2 V(z,t)}{\partial z^2} = lc\frac{\partial^2 V(z,t)}{\partial t^2} \qquad (1)$$

We will define

$$\boxed{u = \frac{1}{\sqrt{lc}} \quad \text{[m/s]}}$$

so that

$$\frac{\partial^2 V(z,t)}{\partial z^2} = \frac{1}{u^2}\frac{\partial^2 V(z,t)}{\partial t^2} \qquad (2)$$

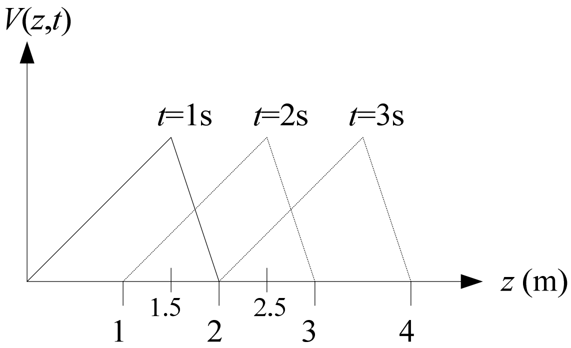
There are two general solutions to (2):

1. $V(z,t) = V^+\left(t - \frac{z}{u}\right)$. (3)

   V^+ is any function containing t, z, and u in the form of the argument shown. It can be verified that (3) is a solution to (2) by substituting (3) into (2) and showing that the LHS equals the RHS.

Equation (3) represents a wave traveling in the +z direction with **speed** $u = 1/\sqrt{lc}$ m/s.

To see this, consider the example below with u = 1 m/s:



At t = 1 s, focus on the peak located at z = 1.5 m. Then,

$$s_+ \equiv t - \frac{z}{u} = 1 - \frac{1.5}{\cancel{u}^1} = -0.5$$

The argument s_+ stays constant for varying t and z if we wish to observe the same position of the function in space as time evolves. At t = 2 s, for example, then

$$s_+ = -0.5 = \cancel{t}^2 - \frac{z}{\cancel{u}^1}$$

Therefore,

$$z = 2.5 \text{ m}$$

So the peak has now moved to position z = 2.5 m at t = 2 s.

Likewise, every point on this function moves the same distance (1 m) in this time (1 s). This is called wave motion.

The speed of this movement is

$$\frac{\Delta z}{\Delta t} = \frac{1}{1} = 1 = u$$

2. $V(z,t) = V^{-}\left(t + \frac{z}{u}\right)$. (4)

This is the second general solution to (2). This function V^{-} represents a wave moving in the -z direction with speed u.

The complete solution to the wave equation (2) is the sum of (3) and (4)

$$V(z,t) = \underbrace{V^{+}\left(t - \frac{z}{u}\right)}_{+z\text{ wave}} + \underbrace{V^{-}\left(t + \frac{z}{u}\right)}_{-z\text{ wave}} \tag{5}$$

V^{+} and V^{-} can be any (suitable differentiable) functions, but with the arguments as shown.

Current Wave Equation Solutions

A similar analysis can be performed for the current on the TL. From equation (10) in the previous lecture

$$\frac{\partial^2 I(z,t)}{\partial z^2} = \frac{1}{u^2}\frac{\partial^2 I(z,t)}{\partial t^2} \tag{6}$$

The complete general solution to this current wave equation can be determined in a manner similar to above as

$$I(z,t)=\underbrace{I^+\left(t-\frac{z}{u}\right)}_{+z\text{ wave}}+\underbrace{I^-\left(t+\frac{z}{u}\right)}_{\text{-}z\text{ wave}} \tag{7}$$

However, the function I^+ can be related to the function V^+, and I^- can be related to V^-. For example, substituting $I^+(t-z/u)$ and $V^+(t-z/u)$ into equation (6) from the last lecture:

$$\frac{\partial I(z,t)}{\partial z}=-c\frac{\partial V(z,t)}{\partial t}$$

(which is one of the telegrapher's equations), then using the chain rule of differentiation

$$\frac{d}{dx}\left[f(v)\right]=\frac{d}{dv}\left[f(v)\right]\cdot\frac{dv}{dx}$$

and differentiating wrt *z* and *t*, then integrating wrt $t-z/u$ gives

$$-\frac{1}{u}I^+=-cV^+$$

or

$$I^+=ucV^+ \tag{8}$$

But,

$$uc=\frac{1}{\sqrt{lc}}c=\sqrt{\frac{c}{l}}$$

We will define

$$\boxed{R_c\equiv\sqrt{\frac{l}{c}}\quad[\Omega]} \tag{9}$$

as the characteristic resistance of the transmission line with units of Ω. (Note that in many texts and reference books, R_c is denoted as Z_0, the characteristic impedance of the TL.)

With (9), equation (8) can be written as

$$I^+\left(t-\frac{z}{u}\right)=\frac{V^+\left(t-\frac{z}{u}\right)}{R_c} \tag{10}$$

Similarly, it can be shown that

$$I^-\left(t+\frac{z}{u}\right)=-\frac{V^-\left(t+\frac{z}{u}\right)}{R_c} \tag{11}$$

The minus sign results since the current is in the -z direction.

Finally, substituting (10) and (11) into (7) gives

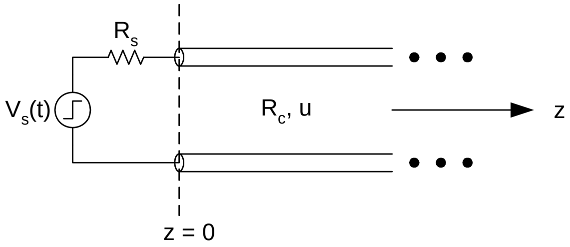
$$\boxed{I(z,t)=\frac{1}{R_c}V^+\left(t-\frac{z}{u}\right)-\frac{1}{R_c}V^-\left(t+\frac{z}{u}\right)} \tag{12}$$

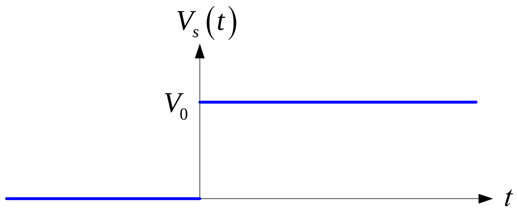
This equation as well as (5)

$$\boxed{V(z,t)=V^+\left(t-\frac{z}{u}\right)+V^-\left(t+\frac{z}{u}\right)} \tag{5}$$

are the **general wave solutions for *V* and *I* on a transmission line**.

Example N12.1: A semi-infinite TL is excited with a unit-step voltage source $V_s(t)=V_0u(t)$ as shown below. Determine the voltage and current on the TL assuming the TL was initially "uncharged" (i.e., $V = I = 0$ everywhere).





At $t = 0$, the source voltage jumps from 0 to V_0 volts. Voltage and current disturbances will then begin propagating down the TL at $t = 0$.

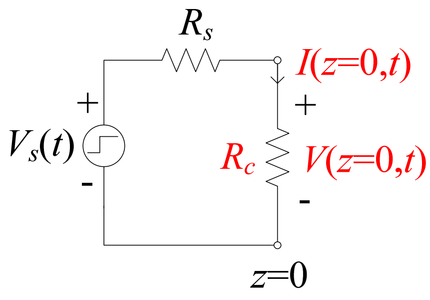
Since there is no termination of the TL (it is semi-infinitely long), the voltage and current waves travel only in the +z direction. Consequently, $V^- = I^- = 0$. So, from (5) and (12)

$$V(z,t) = V^+\left(t - \frac{z}{u}\right) \quad \text{and} \quad I(z,t) = \frac{1}{R_c}V^+\left(t - \frac{z}{u}\right).$$

From (10), the ratio of these voltage and current amplitudes is

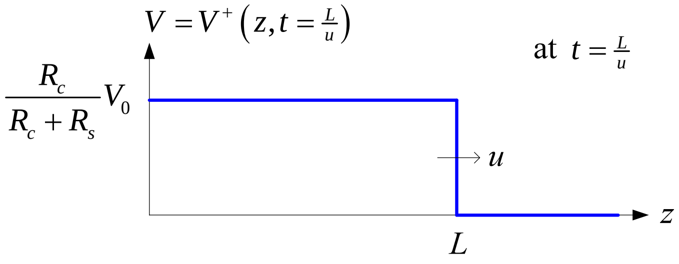
$$\frac{V^+}{I^+} = R_c$$

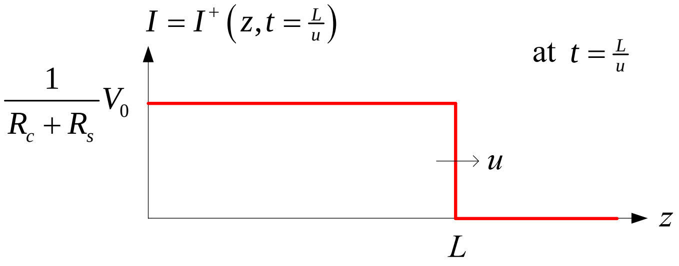
Because $V(z=0,t) = V^+$ and $I(z=0,t) = I^+$, then the equivalent circuit at the input for $t \geq 0$ is



Hence, $$V(0,t) = \frac{R_c}{R_c + R_s} V_s(t) = \begin{cases} 0 & t < 0 \\ \dfrac{R_c}{R_c + R_s} V_0 & t > 0 \end{cases} \tag{13}$$

These voltage and current disturbances propagate with speed u with no attenuation (lossless TLs):

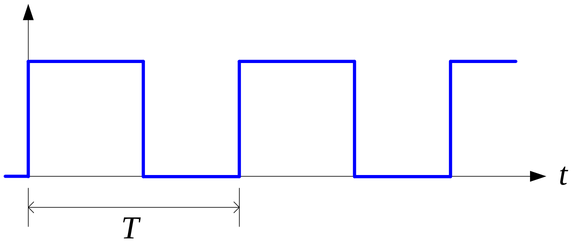




Remember that this transmission line is a model of the wave propagation for any geometry that supports a TEM wave.

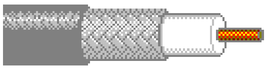
Consider, for example, an RG-58A/U coaxial cable where $R_c = 53.5$ Ω and $u = 69.5\%$ (of c_0). That is, $u = 0.695 \cdot 2.998 \times 10^8 = 2.084 \times 10^8$ m/s.

In 1 ns, the leading edge propagates a distance $L = u \cdot 10^{-9} = 20.8$ cm. Is 1 ns an incredibly short time in electrical circuits? Not really. Consider a 1 GHz clock in a PC:



$$T = \frac{1}{f} = \frac{1}{10^9} = 1.0 \text{ ns!}$$

82240 Coax - RG-58A/U Type



For more information please call

1-800-Belden1

See Put-ups and Colors

Description:

RG-58A/U type, 20 AWG solid .033" bare copper conductor, plenum, FEP teflon insulation, tinned copper braid shield (95% coverage), Flamarrest® jacket.

PHYSICAL CHARACTERISTICS:

CONDUCTOR:

Number of Coax	1
Total Number of Conductors	1
RG Type	58/U
AWG	20
Stranding	Solid
Conductor Diameter	.032 in.
Conductor Material	BC - Bare Copper

INSULATION:

Insulation Material Trade Name	Teflon®
Insulation Material	FEP - Fluorinated Ethylene Propylene
Insulation Diameter	.107 in.

OUTER SHIELD:

Outer Shield Type	Braid
Outer Shield Material	TC - Tinned Copper
Outer Shield %Coverage	95 %

OUTER JACKET:

Outer Jacket Material Trade Name	Flamarrest®
Outer Jacket Material	LS PVC - Low Smoke Polyvinyl Chloride

OVERALL NOMINAL DIAMETER:

Overall Nominal Diameter	.159 in.

MECHANICAL CHARACTERISTICS:

Operating Temperature Range	0°C To +75°C
UL Temperature Rating	75°C
Bulk Cable Weight	30 lbs/1000 ft.
Max. Recommended Pulling Tension	47 lbs.

Min. Bend Radius (Install)	2 in.

APPLICABLE SPECIFICATIONS AND AGENCY COMPLIANCE:

APPLICABLE STANDARDS:

NEC/(UL) Specification	CMP
CEC/C(UL) Specification	CMP
EU CE Mark (Y/N)	No
EU RoHS Compliant (Y/N)	Yes
EU RoHS Compliance Date (mm/dd/yyyy):	10/01/2005

FLAME TEST:

UL Flame Test	NFPA 262
C(UL) Flame Test	FT6

PLENUM/NON-PLENUM:

Plenum (Y/N)	Y
Non-Plenum Number	8240

ELECTRICAL CHARACTERISTICS:

Nom. Characteristic Impedance	53.5 Ohms
Nom. Inductance	0.070 µH/ft
Nom. Capacitance Conductor to Shield	26.4 pF/ft
Nominal Velocity of Propagation	69.5 %
Nominal Delay	1.48 ns/ft
Nom. Conductor DC Resistance @ 20 Deg. C	10.2 Ohms/1000 ft
Nominal Outer Shield DC Resistance @ 20°C	6.7 Ohms/1000 ft

Nom. Attenuation :

Description	**Frequency (MHz)**	**Start Frequency (MHz)**	**Stop Frequency (MHz)**	**Nom. Attenuation (dB/100 ft.)**
	1			0.5
	10			1.2
	50			3.0
	100			4.3
	200			6.4
	400			9.7
	700			13.7
	900			16.1
	1000			17.3

Nom. Power Rating :

82240 Coax - RG-58A/U Type

Description	Frequency (MHz)	Start Frequency (MHz)	Stop Frequency (MHz)	Nom. Power Rating (W)
	1			4798.7
	10			1937.2
	50			782.4
	100			542.2
	200			370.3
	400			248.5
	700			177.7
	900			153.0
	1000			143.2

Max. Operating Voltage - UL	300 V RMS
Max. Operating Voltage - Non-UL	1400 V RMS

NOTES:

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PUT-UPS AND COLORS:

Item	Description	Put-Up (ft.)	Ship Weight (lbs.)	Jacket Color	Notes
82240 8771000	#20 FEP RG-58/U FLMRST	1000	24	NATURAL	C
82240 877U1000	#20 FEP RG58/U FLMRST	U1000	26	NATURAL	
82240 877U500	#20 FEP RG58/U FLMRST	U500	13.5	NATURAL	

C = CRATE REEL PUT-UP.

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