## Chapter 18 – Fourier Transform (FT)

Chapter is mathematically “heavy”.

We will use Tables & calculators.

Focus on F-T applications in Circuits.

The FT represents with integrals non-periodic functions,
as Fourier Series summations (Σ) represent periodic functions.

**The FT extends the concept of a frequency spectrum to nonperiodic functions,** by assuming that a nonperiodic function is periodic with $T=\infty$.

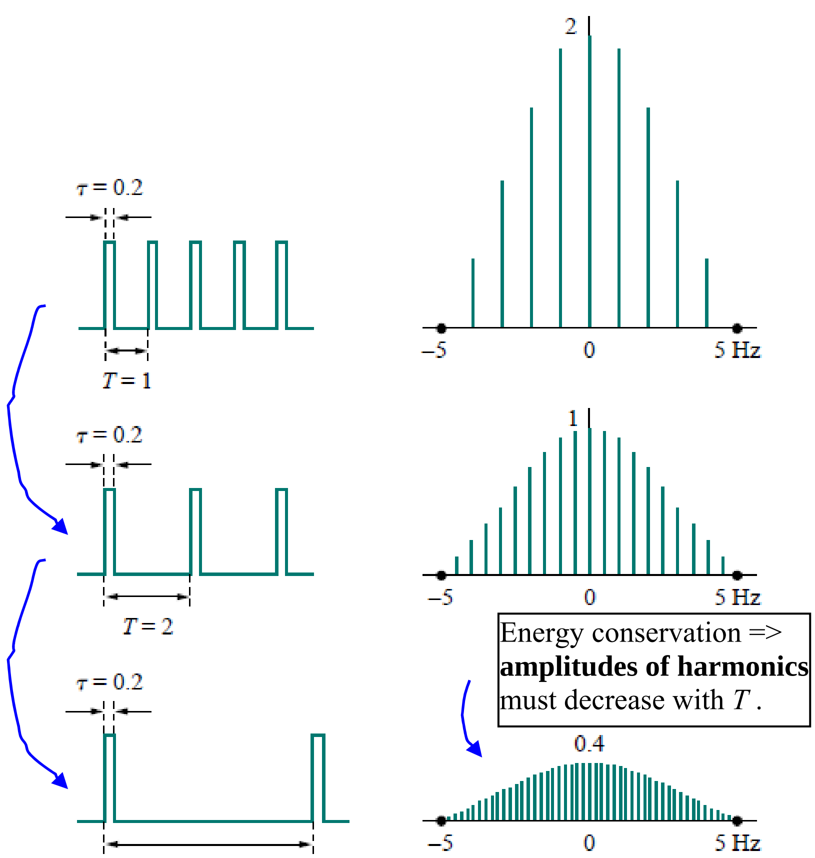
FT:

**The FT is an integral transformation of $f(t)$ from T-D → F-D.**

IFT:

f (t) and F(ω) form the Fourier transform pairs: $f(t) <=> F(\omega)$.

As $T\uparrow$, the discrete spectrum ultimately becomes continuous




Effect of increasing $T$ on the spectrum of the periodic pulse trains in Fig. 17.1(b).

Properties of FT: Quite similar to Laplace with $s \rightarrow j\omega$ – see eqs in textbook.

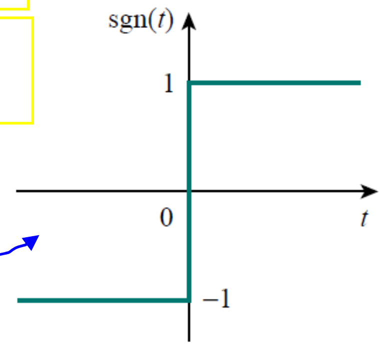
Time-Scaling, Frequency Shifting (or Amplitude Modulation, AM), …

TABLE | Properties of the Fourier transform.

| Property | $f(t)$ | $F(\omega)$ |
|---|---|---|
| Linearity | $a_1 f_1(t) + a_2 f_2(t)$ | |
| Scaling | $f(at)$ | $\frac{1}{\lvert a\rvert} F\left(\frac{\omega}{a}\right)$ |
| Time shift | $f(t-a)u(t-a)$ | $e^{-j\omega a} F(\omega)$ |
| Frequency shift | $e^{j\omega_0 t} f(t)$ | |
| Modulation | $\cos(\omega_0 t)\ f(t)$ | |
| Time differentiation | $\frac{df}{dt}$ | |
| | $\frac{d^n f}{dt^n}$ | |
| Time integration | $\int_{-\infty}^{t} f(t)\, dt$ | $\frac{F(\omega)}{j\omega} + \pi F(0)\, \delta(\omega)$ |
| Frequency differentiation | $t^n f(t)$ | $(j)^n \frac{d^n}{d\omega^n} F(\omega)$ |
| Reversal | $f(-t)$ | $F(-\omega)$ or $F^*(\omega)$ |
| Duality | $F(t)$ | $2\pi f(-\omega)$ |
| Convolution in $t$ | $f_1(t) * f_1(t)$ | $F_1(\omega) F_2(\omega)$ |
| Convolution in $\omega$ | $f_1(t) f_1(t)$ | $\frac{1}{2\pi} F_1(\omega) * F_2(\omega)$ |

## TABLE 2 Fourier transform pairs.

| $f(t)$ | $F(\omega)$ |
|---|---|
| $\delta(t)$ | |
| $1$ | |
| $u(t)$ | |
| $u(t+\tau)-u(t-\tau)$ | $2\dfrac{\sin\omega\tau}{\omega}$ |
| $\lvert t\rvert$ | $\dfrac{-2}{\omega^2}$ |
| $\text{sgn}(t)$ | $\dfrac{2}{j\omega}$ |
| $e^{-at}u(t)$ | $\dfrac{1}{a+j\omega}$ |
| $e^{at}u(-t)$ | $\dfrac{1}{a-j\omega}$ |
| $t^n e^{-at}u(t)$ | $\dfrac{n!}{(a+j\omega)^{n+1}}$ |
| $e^{-a\lvert t\rvert}$ | $\dfrac{2a}{a^2+\omega^2}$ |
| $e^{j\omega_0 t}$ | $2\pi\delta(\omega-\omega_0)$ |
| $\sin\omega_0 t$ | |
| $\cos\omega_0 t$ | |
| $e^{-at}\sin\omega_0 t\,u(t)$ | $\dfrac{\omega_0}{(a+j\omega)^2+\omega_0^2}$ |
| $e^{-at}\cos\omega_0 t\,u(t)$ | $\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$ |




The signum function

$\text{sgn}(t) = f(t) = u(t) - u(-t)$

Ex. 18.1: Find the FT of:

a) $\delta(t-t_0)$

Def => $F\{\delta(t)\} = 1.$ δ(t) consists equally of all frequencies. (the magnitude of the spectrum of δ(t) is constant.)

b) $e^{j\omega_0 t}$

Def / Shifting Property => $F\{e^{j\omega_0 t}\} = 2\pi\delta(\omega - \omega_0)$

E.g. If we set $\omega_0 = 0$ => $F[1] = 2\pi\delta(\omega)$

c) $\cos(\omega_0 t)$

Using b) =>

Plot:

(notice the negative harmonic frequencies, as expected by the exponential Fourier Series of periodic functions).

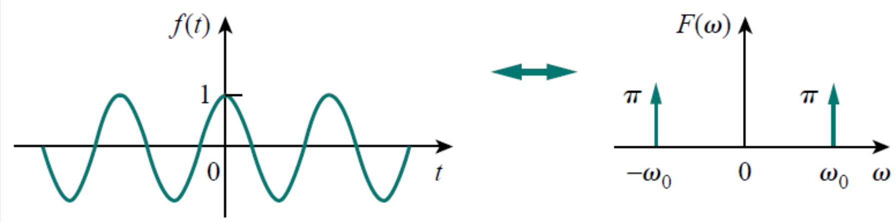



Figure Fourier transform of $f(t) = \cos\omega_0 t$.

[ Negative frequencies are simply a result of the way the mathematical analysis is structured. The positive and negative parts of the spectrum combine to produce a real function @ $f > 0$. If the negative frequency was not there, the time function would contain an imaginary part.

A *complex sinusoid* preserves the distinction between positive and negative ω. Both frequencies are present, as implied by the inverse of Euler's formula: $\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$.

Every real sinusoid consists of an equal contribution of positive and negative frequency components. This is true of all real signals. In spectrum analysis, every real signal contains equal amounts of positive and negative frequencies, i.e., if $X(\omega)$ denotes the spectrum of the real signal $x(t)$, we will always have $|X(-\omega)| = |X(\omega)|$.

Note that, mathematically, the complex sinusoid $Ae^{j(\omega t+\phi)}$ is really simpler and more basic than the real sinusoid $A\sin(\omega t+\phi)$ because $e^{j\omega t}$ consists of one frequency ω while sin(ωt) really consists of two frequencies ω and –ω. We may think of a real sinusoid as being the sum of a positive-frequency and a negative-frequency complex sinusoid, so in that sense real sinusoids are ``twice as complicated'' as complex sinusoids.

Ex. 18.2: Find FT of a single rect. pulse of width = $\tau$ and height = $A$

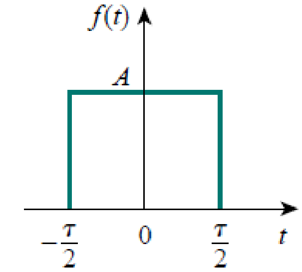



Figure A rectangular pulse;

(See math in book) => ...

=>

Compare to Fourier Series of pulse train (Ch. 17):

⇨ F(ω) is continuous & Envelope of the FS

⇨ E.g.: If A=10 & τ=2 =>

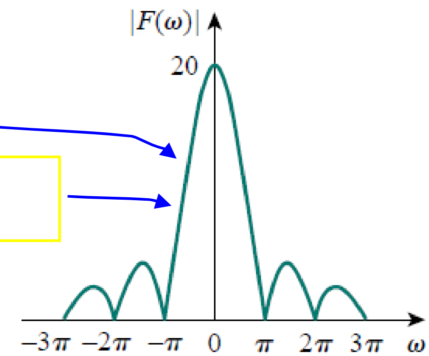
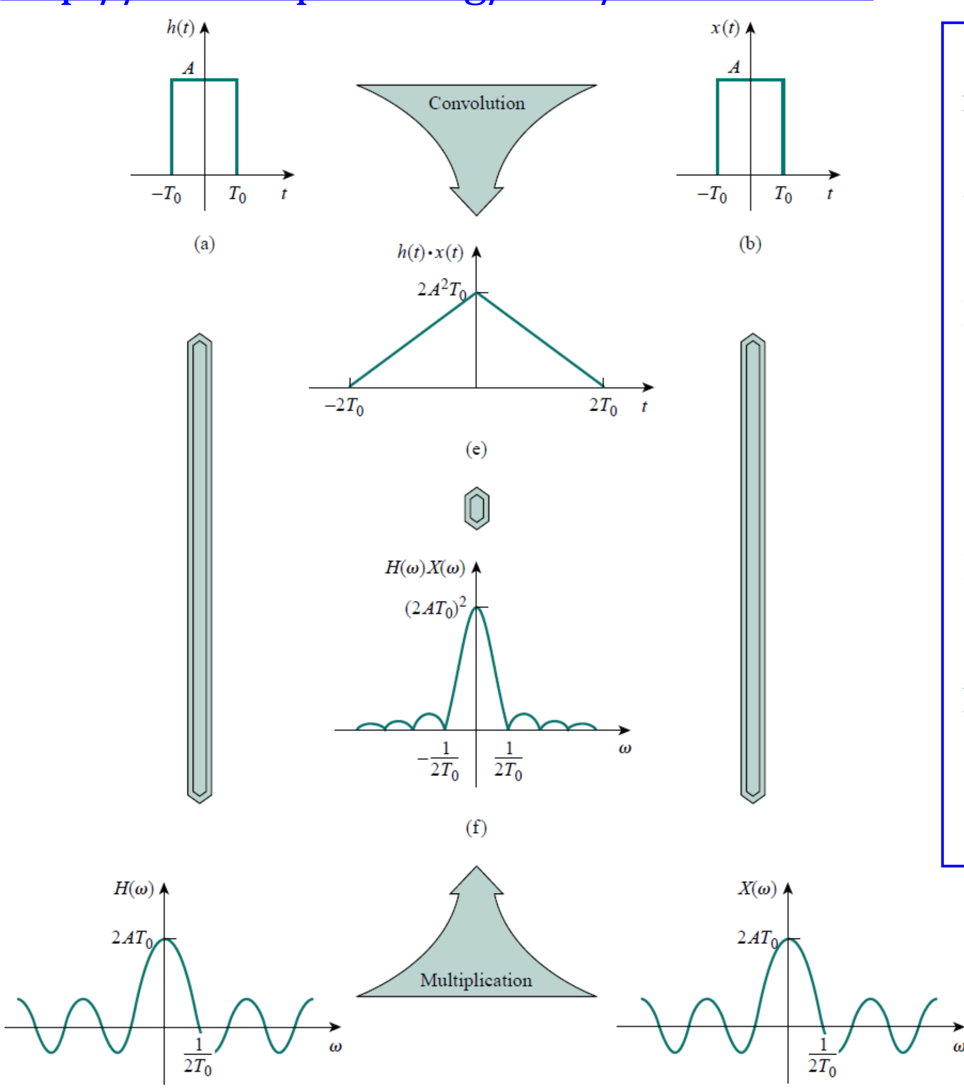



Figure Amplitude spectrum of the rectangular pulse

Convolution:

[http://en.wikipedia.org/wiki/Convolution](http://en.wikipedia.org/wiki/Convolution)




Convolution is a mathematical way of combining 2-signals to form a 3rd-signal.

It relates the 3 signals of interest:
- Input,
- Output, and
- Impulse response.

It is the most important technique in DSP because it helps describe systems by their *impulse response*.

Circuit Applications of FT:

The FT generalizes the phasor technique to non-periodic functions. So we apply FTs to circuits with nonsinusoidal excitations (same as with sinusoidal).

*Note*: Fourier analysis cannot handle circuits with initial conditions. It is:

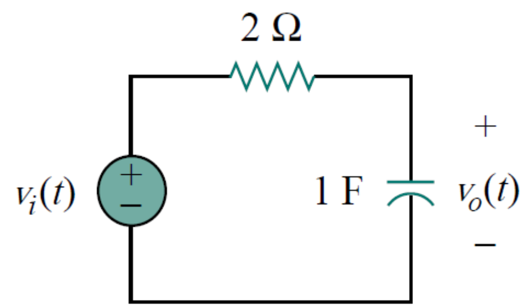
- Identical to Laplace but with $s = j\omega$
- Ohm's law in FD: $V(\omega) = Z(\omega)\, I(\omega)$
- If the input $x(t) = \delta(t)$ =>

$X(\omega) = 1$ => (since $Y = H \cdot X$)

=> output = $Y(\omega) = H(\omega) = \mathcal{F}\{h(t)\}$ [so $H(\omega)$ is the FT of the imp. resp. $h(t)$.]

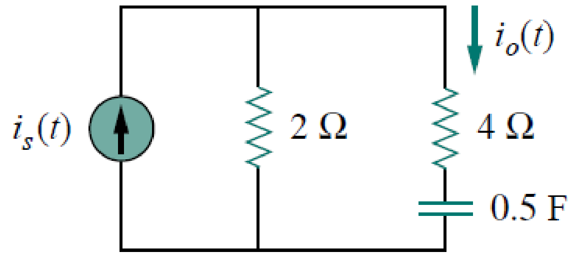
**Example 18.7.**

Find $v_0(t)$ in the circuit for $v_i(t) = 2e^{-3t}u(t)$.



<u>Example 18.8.</u>

Using the FT method,
find $i_0(t)=?$ when $i_s(t) = 10 \sin 2t$ A.



Homework:

Example 18.8 using Laplace.
PP 18.7
PP 18.8