Chapter 17 – Fourier Series

Since a periodic function repeats every T seconds, we can also write:

$$f(t) = f(t + nT) \qquad (16.1)$$, n integer, T period of f.

Any (nonsinusoidal) practical periodic function $f(t)$ of frequency ω_0 can be expressed as an infinite sum of *sin* or *cos* functions that are integral multiples of ω_0:

$f(t) =$ (16.2)

or

→ $\omega_0 = 2\pi/T$ -

sin $n\omega_0 t$ -

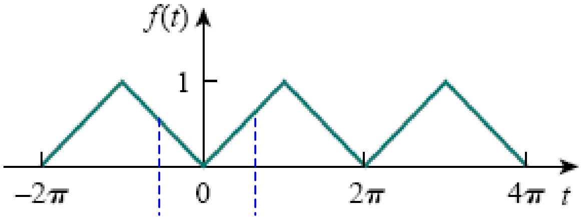
n=odd → *odd harmonic*

n=even → *even harmonic*

a_n and b_n = *Fourier coefficients*

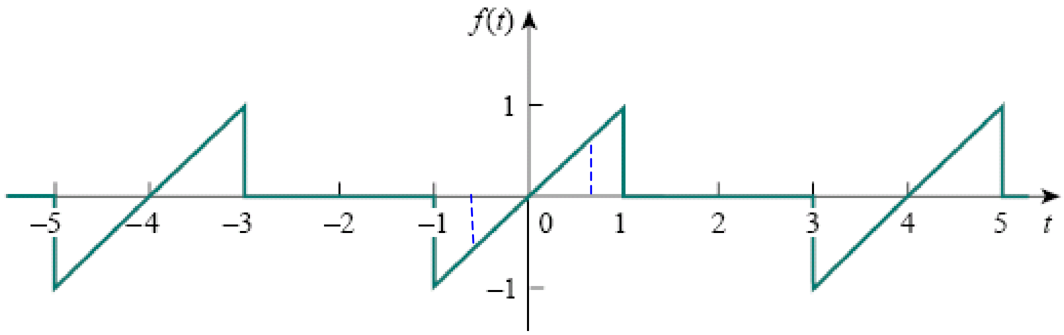
Symmetry: Helps in simplifying the analysis

E.g.



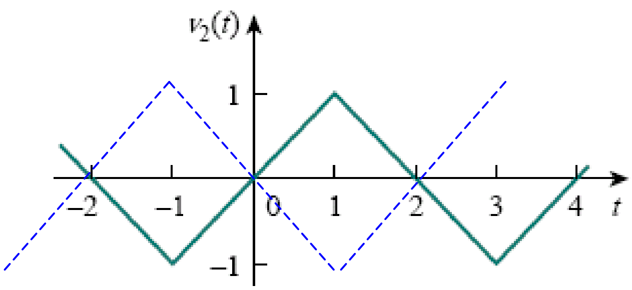
[flip about]

E.g.



[flip about]

E.g.



[slide it and flip about]

We can determine the *Fourier coefficients* α_0, α_n, and b_n with *Fourier analysis* and the following integrals (proofs are in the book):

and with the help of:

$$\int \cos at\, dt = \frac{1}{a}\sin at \qquad (16.15a)$$

$$\int \sin at\, dt = -\frac{1}{a}\cos at \qquad (16.15b)$$

$$\int t\cos at\, dt = \frac{1}{a^2}\cos at + \frac{1}{a}t\sin at \qquad (16.15c)$$

$$\int t\sin at\, dt = \frac{1}{a^2}\sin at - \frac{1}{a}t\cos at \qquad (16.15d)$$

and:

$$\int_0^T \sin n\omega_0 t\, dt = 0 \qquad (16.4a)$$

$$\int_0^T \cos n\omega_0 t\, dt = 0 \qquad (16.4b)$$

$$\int_0^T \sin n\omega_0 t \cos m\omega_0 t\, dt = 0 \qquad (16.4c)$$

$$\int_0^T \sin n\omega_0 t \sin m\omega_0 t\, dt = 0, \qquad (m \neq n) \qquad (16.4d)$$

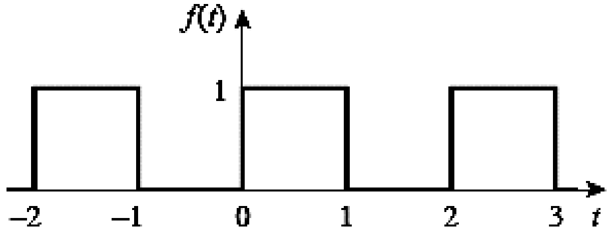
$$\int_0^T \cos n\omega_0 t \cos m\omega_0 t\, dt = 0, \qquad (m \neq n) \qquad (16.4e)$$

$$\int_0^T \sin^2 n\omega_0 t\, dt = \frac{T}{2} \qquad (16.4f)$$

$$\int_0^T \cos^2 n\omega_0 t\, dt = \frac{T}{2} \qquad (16.4g)$$

Ex. 16.1:

Determine the Fourier series of the *square wave* waveform.



Sol:

Express one period

Observations:

Symmetry:

Then write f(t) as a Fourier series (16.3):

$$f(t) = \underbrace{a_0}_{\text{dc}} + \underbrace{\sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}_{\text{ac}} \quad (16.3)$$

Must determine coefficients a_0, a_n, b_n and ω_0.

1. Period: T=2 sec => $\omega_0=2\pi f = 2\pi/T$ => $\omega_0=\pi$.

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$ is also the *average*.

So, $a_0 = ½$

For a_0 we could also solve the $\int$:

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\left[\int_0^1 1\,dt + \int_1^2 0\,dt\right] = \frac{1}{2}t\Big|_0^1 = \frac{1}{2} \qquad (16.1.3)$$

Then, we can find a_n and b_n from:

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt \qquad (16.8)$$

$$= \frac{2}{2}\left[\int_0^1 1\cos n\pi t\,dt + \int_1^2 0\cos n\pi t\,dt\right] \qquad (16.1.4)$$

$$= \frac{1}{n\pi}\sin n\pi t\,\Big|_0^1 = \frac{1}{n\pi}\sin n\pi = 0$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt \qquad (16.9)$$

$$= \frac{2}{2}\left[\int_0^1 1\sin n\pi t\,dt + \int_1^2 0\sin n\pi t\,dt\right]$$

$$= -\frac{1}{n\pi}\cos n\pi t\,\Big|_0^1 \qquad (16.1.5)$$

$$= -\frac{1}{n\pi}(\cos n\pi - 1), \qquad \cos n\pi = (-1)^n$$

$$= \frac{1}{n\pi}[1 - (-1)^n] = \begin{cases} \dfrac{2}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

Now substitute all results in: $$f(t) = \underbrace{a_0}_{\text{dc}} + \underbrace{\sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}_{\text{ac}} \qquad (16.3)$$

to get the *Fourier series*:

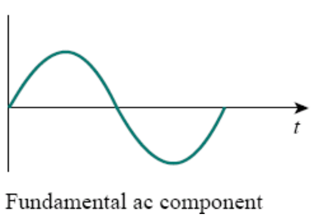
Notice, only odd harmonics (3^{rd}, 5^{th}, …etc), so can write:

[*k*=integer]

$$f(t) = \frac{1}{2} \quad \rightarrow$$

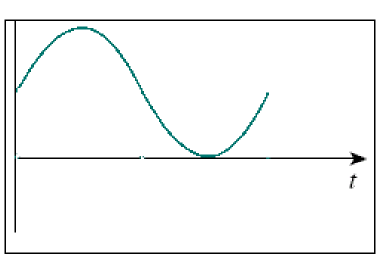
$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sin \pi t$$

\+



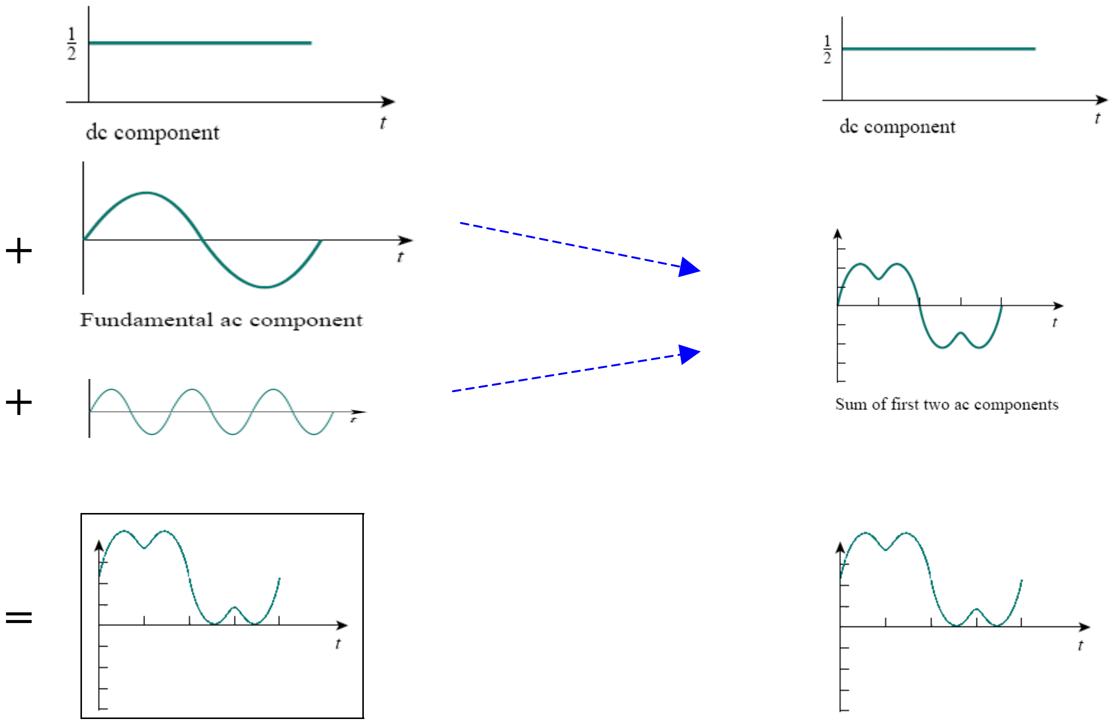
(ampl = $2/\pi$)

=



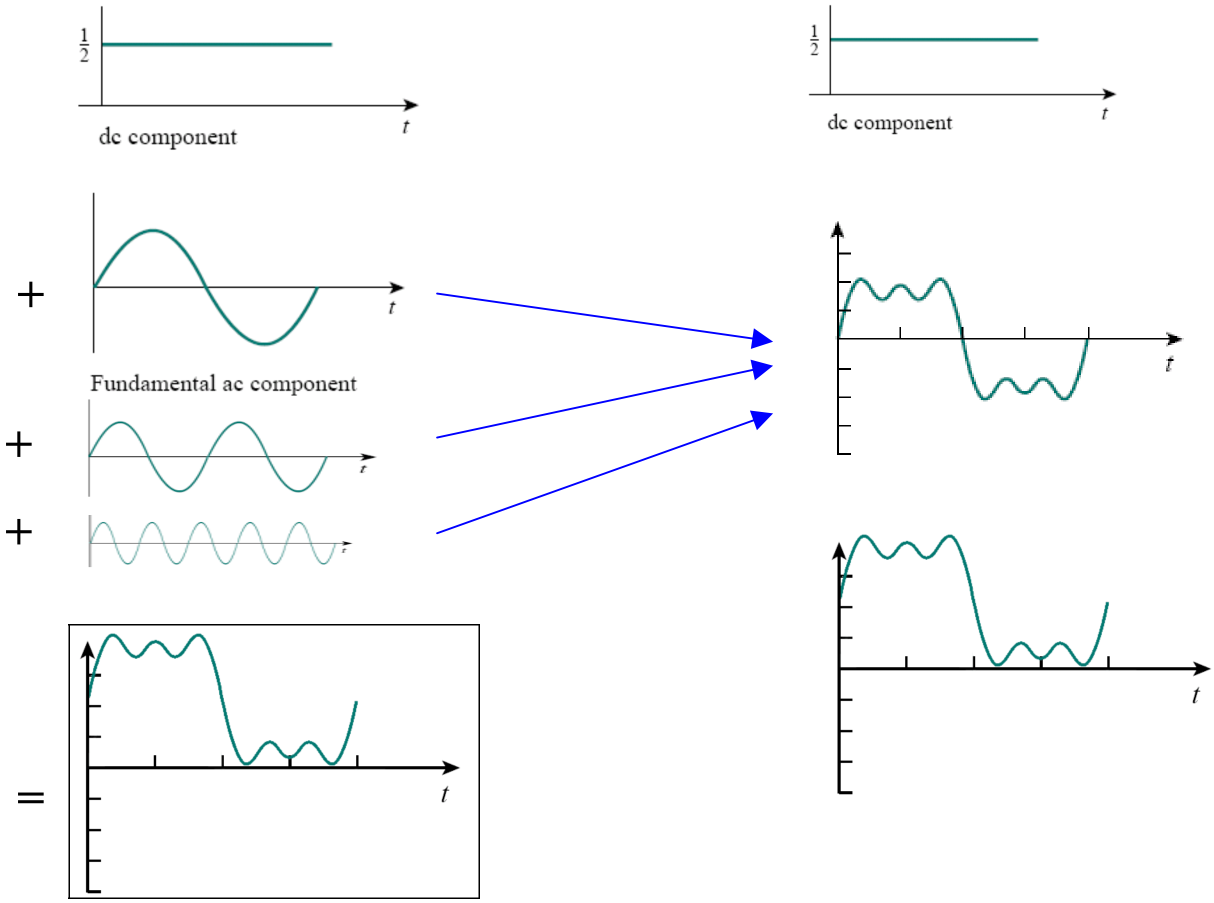
best approximation with 1 term that we can make.

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sin \pi t + \frac{2}{3\pi}\sin 3\pi t$$ [2 harmonics]

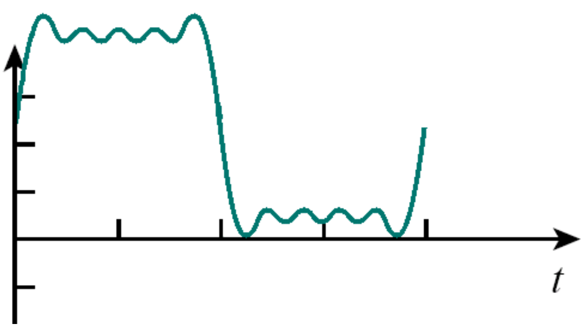


best approximation with 2 terms that we can make

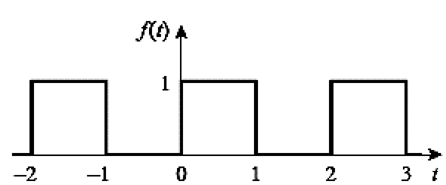
$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sin\pi t + \frac{2}{3\pi}\sin 3\pi t + \frac{2}{5\pi}\sin 5\pi t \qquad \text{[3 harmonics]}$$



With the first 4 harmonics we would obtain:



… to keep approaching the original:



We found earlier the Fourier series of the given square wave:

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}\sin n\pi t, \qquad n = 2k-1 \qquad (16.1.7)$$

[k=integer]

What about the signal's spectrum?

The *frequency spectrum* of a signal consists of the plots of the amplitudes and phases of the harmonics versus frequency

We obtain these spectra by expressing the given Fourier series (16.3) …

$$f(t) = \underbrace{a_0}_{\text{dc}} + \underbrace{\sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}_{\text{ac}} \qquad (16.3)$$

… with an amplitude and phase as:

To do that, we apply to the Σ of (16.10) the identity:

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

with which (16.10) becomes:

$$a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n) \quad \rightarrow$$

$$= a_0 + \sum_{n=1}^{\infty} (A_n \cos\phi_n) \cos n\omega_0 t - (A_n \sin\phi_n) \sin n\omega_0 t \qquad (16.12)$$

Equate the *sin* and *cos* terms in (16.3) & (16.12) =>

$$a_n = A_n \cos\phi_n, \qquad b_n = -A_n \sin\phi_n \tag{16.13a}$$

or

$$A_n = \sqrt{a_n^2 + b_n^2}, \qquad \phi_n = -\tan^{-1}\frac{b_n}{a_n} \tag{16.13b}$$

and we may also write: $$A_n \angle \phi_n = a_n - jb_n \tag{16.14}$$

We will use these eqs to find the *frequency spectrum*.

In our example we found:

$a_n = 0$, so $$A_n = \qquad = \begin{cases} \dfrac{2}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases} \tag{16.1.8}$$

and $$\phi_n = \qquad = \begin{cases} -90^\circ, & n = \text{odd} \\ 0, & n = \text{even} \end{cases} \tag{16.1.9}$$

Plot A_n and φ_n for different values of $n\omega_0$.

$n\omega_0 = n\pi$ => Plot for different n = 0, 1, 2, 3, … :

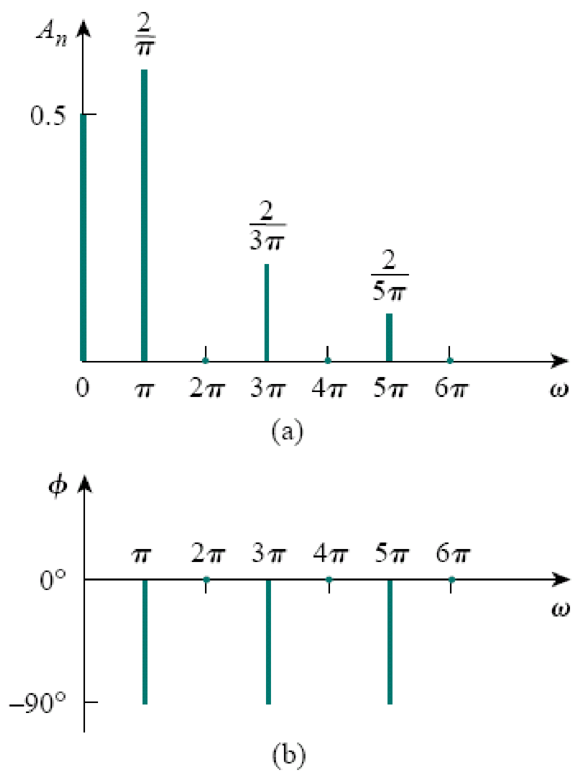


Figure (a) Amplitude and (b) Phase spectrum of the square wave function

Gibbs phenomenon:

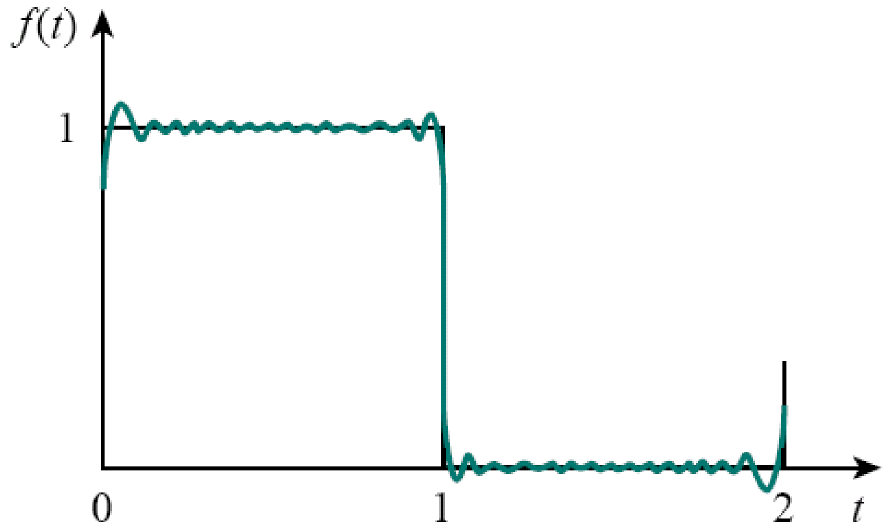
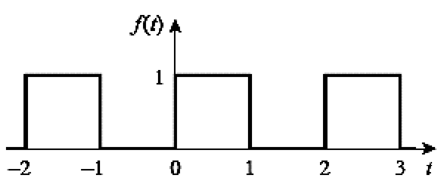


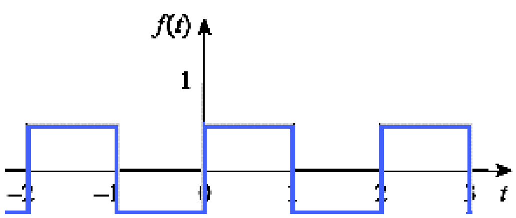
Figure 16.3 Truncating the Fourier series at $N = 11$; Gibbs phenomenon.

Symmetry: Use when possible! It simplifies the terms of the series. [book]

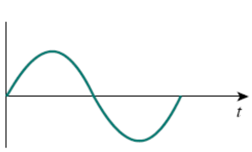
If instead of :



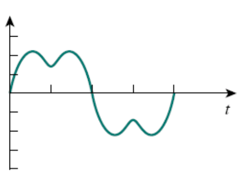
we had :



=> Odd symmetry =>



Fundamental ac component ,



Sum of first two ac components , … can be made only with *sin* terms.
All cosines shall disappear.

- Is the Fourier representation *approximate* or *absolutely equal*?

Signal generators:

Cannot give exactly a *square wave* cause they cannot give all the really high freqs. So we get an approximation.

The generator would look like it passes the signal from a LPF that cuts the higher freqs (harmonics).

Video: Sine Wave to Square Wave using Fourier Series
http://www.youtube.com/watch?v=y6crWlxKB_E

We found earlier the Fourier series of the given square wave:

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}\sin n\pi t, \qquad n = 2k-1 \qquad (16.1.7)$$

[k=integer]

What about the spectrum of that square wave signal?

The *frequency spectrum* of a signal consists of the plots of the amplitudes and phases of the harmonics versus frequency

Frequency Spectrum.

Amplitude spectrum of *f (t)*: *Phase spectrum* of *f (t)*:

We obtain both spectra by expressing our given Fourier series in (16.3)…

$$f(t) = \underbrace{a_0}_{\text{dc}} + \underbrace{\sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}_{\text{ac}} \qquad (16.3)$$

… with an amplitude A_n and phase φ_n as:

$$f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n) \qquad (16.10)$$

To do that, apply the ID: $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

(16.10) => $$f(t) = a_0 + \sum_{n=1}^{\infty}(A_n\cos\phi_n)\cos n\omega_0 t - (A_n \sin\phi_n)\sin n\omega_0 t \qquad (16.12)$$

Next, equate *sin = cos* terms in (16.3 & 16.12) =>

or solve for

(which are the needed Fourier amplitude and phase).

and we may also write: $A_n \angle \phi_n = a_n - jb_n$ (16.14)

where the real part represents the cos terms and the imaginary represents the sin terms.

with these results [$A_n = \sqrt{(a_n^2 + b_n^2)}$, $\varphi_n = -\tan^{-1}\frac{b_n}{a_n}$] (16.13b)

we can find and plot the *frequency spectrum* of the square wave.

For our square pulse we found:

$$\omega_0 = \pi \quad \Rightarrow \quad n\omega_0 = n\pi.$$

$$a_n = 0, \quad b_n = \begin{cases} \dfrac{2}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

So, $A_n =$

and $\phi_n =$

Then, **plot** A_n **and** φ_n **vs** ω for various n = 0, 1, 2, 3,… =>

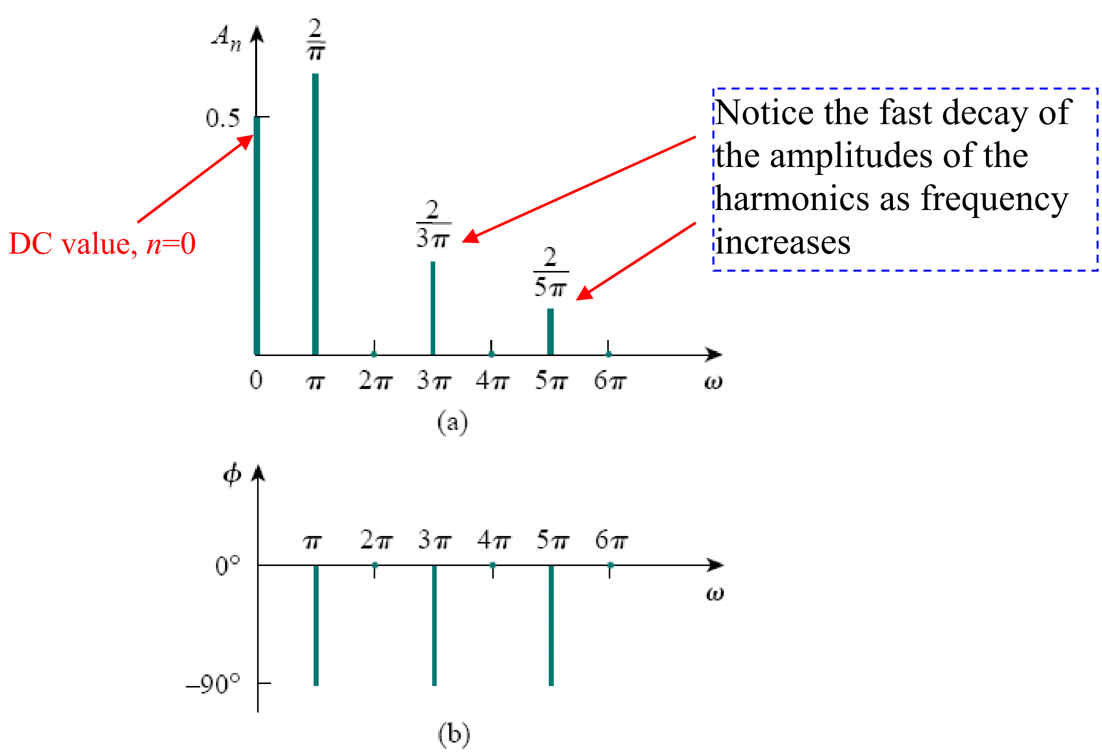


Figure (a) Amplitude and (b) Phase spectrum of the square wave function

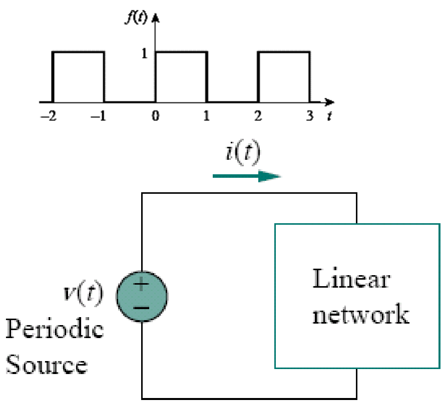
Signal generators:

Do not produce *exactly* a *square wave* (cannot create high freqs).
=> in practice we get an approximation.

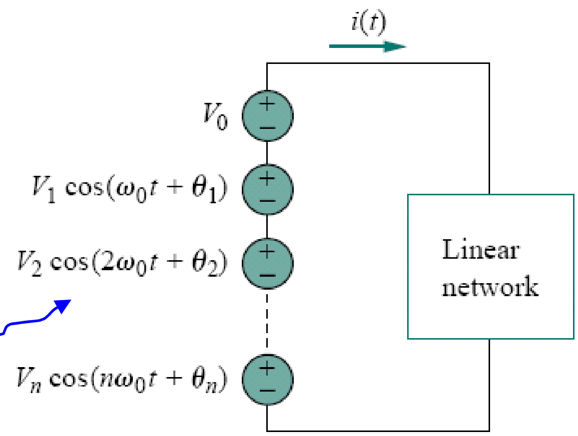
[generator looks like LPF that cuts higher harmonics]

Circuit Applications:

We can create a square pulse as an input,
by using many *sin* sources.



(a)



(b)

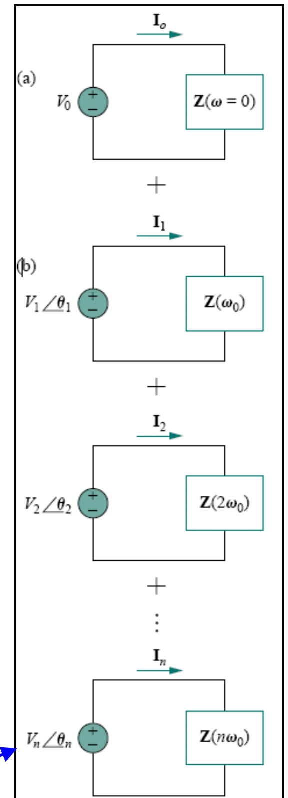


Figure Steady-state responses: (a) dc component, (b) ac component (frequency domain).

a) Linear network excited by a periodic voltage source.
b) Fourier series representation (in T-D).

Generated by mixing the signal with itself:
$\sin^2(\omega t) = ½ – ½ \cos(2\omega t)$ [2× freq!]
(note the 3dB loss of the mixer)

A "Linear Network" is needed for Superposition to work (i.e. made of R, L & C only).

Note: The load impedance $\mathbf{Z}_L$ may change with ω =>
The ckt response will also be different for each ω:

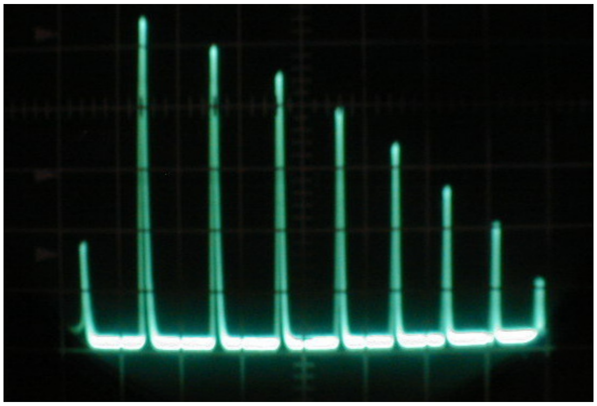
e.g. as $\omega\uparrow$ => Z_L= ____ and Z_C= ____

So, in the lab, to get a square wave,
we need to place the generators in series
and at exactly the correct frequencies.

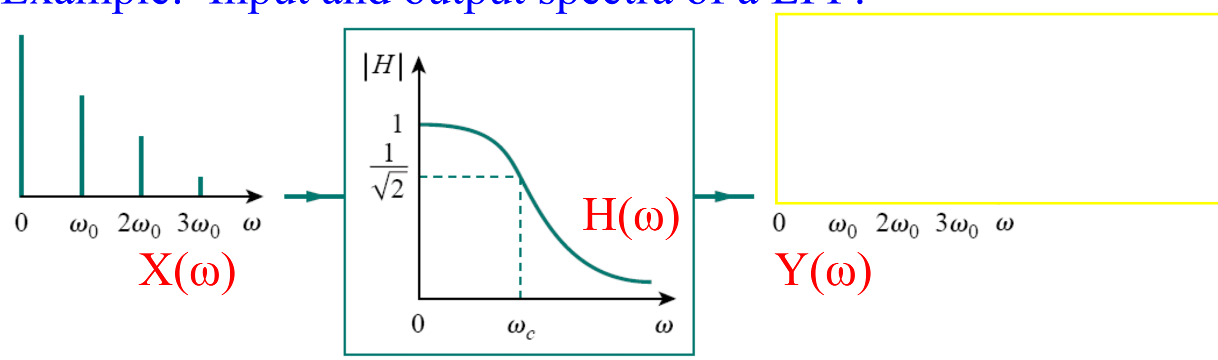
Video: Audio of a Sine Wave to Square Wave, using Fourier Series
http://www.youtube.com/watch?v=y6crWlxKB_E

Spectrum Analyzer:

Shows the spectrum of a waveform (= all the spikes @ each freq (ω)).



Example: Input and output spectra of a LPF:



In F-D, output $Y(\omega)$ = input $\times$ $H(\omega)$.

[Notice the reduced amplitude]

Question: What is the output of a square wave passing through an LPF with $\omega_c \ll \omega_0$?

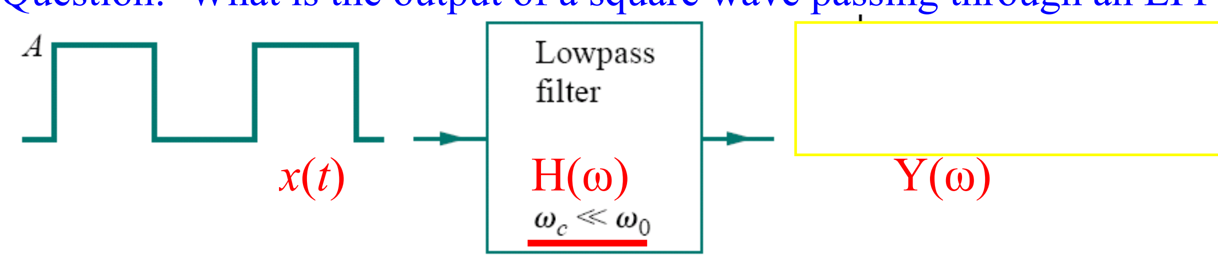
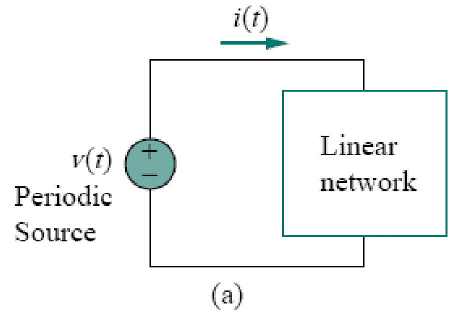


Fig. When $\omega_c << \omega_0$, the LPF passes only the DC component.

In circuits, we apply the Fourier Series in 4 steps:



Note: Source $v(t)$ is periodic, non-sinusoidal
=> has harmonics.

1. Express the excitation as a Fourier series:

e.g. $$v(t) = V_0 + \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t + \theta_n)$$

[Note, the AC component here $\mathbf{V}_n = V_n\angle\theta_n$ has several harmonics.]

We can regard this as a set of series-connected *sin* sources that have their own amplitude and frequency.

2. Transform ckt from T-D → F-D.

3. Find the response of each term (DC & AC components) in the above Fourier series:

a. Find the response to the DC component in F-D or T-D:
- In F-D: set $n = 0$ or $\omega = 0$ (see Fig. 17.19a).
- In T-D: replace L→S.C. and C→O.C.

b. Find the response to the AC component by applying the phasor techniques (Ch. 9) as in Fig. 17.19b:
i.e. Represent the network by its $\mathbf{Z}(n\omega_0)$ or $\mathbf{Y}(n\omega_0)$.

$\mathbf{Z}(n\omega_0)$ = input impedance at the source when ω is everywhere replaced by $n\omega_0$.

[Similarly for $\mathbf{Y}(n\omega_0)$ …]

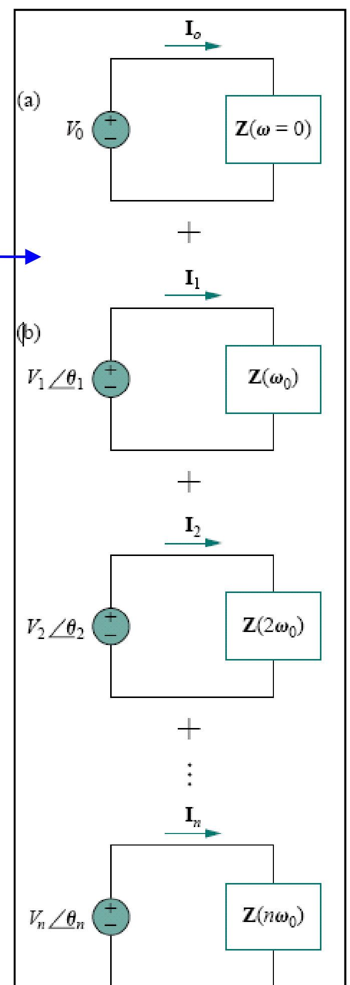


Figure Steady-state responses: (a) dc component, (b) ac component (frequency domain).

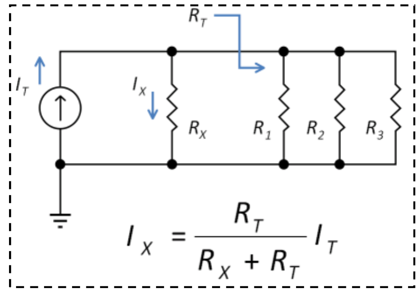
4. Add all individual responses using superposition:

E.g. for the case of Fig. 17.19
the T-D current at the load will be:

$$i(t) = i_0(t) + i_1(t) + i_2(t) + \cdots$$
$$= \mathbf{I}_0 + \sum_{n=1}^{\infty} |\mathbf{I}_n| \cos(n\omega_0 t + \psi_n) \quad (16.41)$$

where each component $\mathbf{I}_n$ (with frequency $n\omega_0$) has been transformed to the T-D to get $i_n(t)$.
ψ_n is the argument of $\mathbf{I}_n$.

Example:



Example: [SKIP]

17.5. Average Power and RMS Values

Similar to Chapter 11,
the average power absorbed by a circuit due to a periodic excitation, is:

The RMS value (=effective value) of a periodic $v(t)$ or $i(t)$ is:

$$F_{rms} = \sqrt{\frac{1}{T}\int_0^T f^2(t)\,dt} \quad = \text{[using the Fourier coeffs } (a_n \,\&\, b_n)\text{]:}$$

or

E.g. the power dissipated in the resistor is: $P = R \cdot I_{rms}^2 = \frac{V_{rms}^2}{R}$

(with $f(t)=i(t)$ or $f(t)=v(t)$ respectively)

Choosing R=1Ω =>

17.6. Exponential Fourier Series

A more compact way to express the Fourier series: The Exponential Form:

Important:

The plots of magnitude & phase of c_n vs $n\omega_0$
comprise the "*complex amplitude spectrum*" of $f(t)$
and the "*complex phase spectrum*" of $f(t)$, respectively.

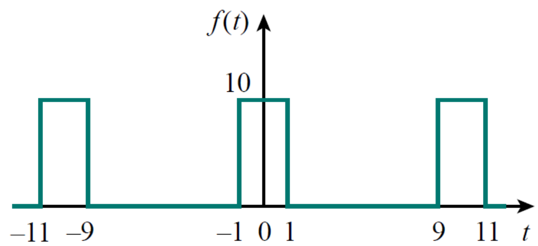
Both spectra together form the "*complex frequency spectrum*" of $f(t)$.

RMS value of periodic signal $f(t)$:

Power dissipated by a R=1Ω:

Power spectrum of $f(t)$ = the plot of $|c_n|^2$ vs. $n\omega_0$.

<u>Example</u>:
Find the amplitude and phase spectra of this periodic pulse train:



<u>Sol</u>:

We know that

=> need find T & ω_0:

The pulse train has: $T = 10$ => $\omega_0 = 2\pi/T = \pi/5$.

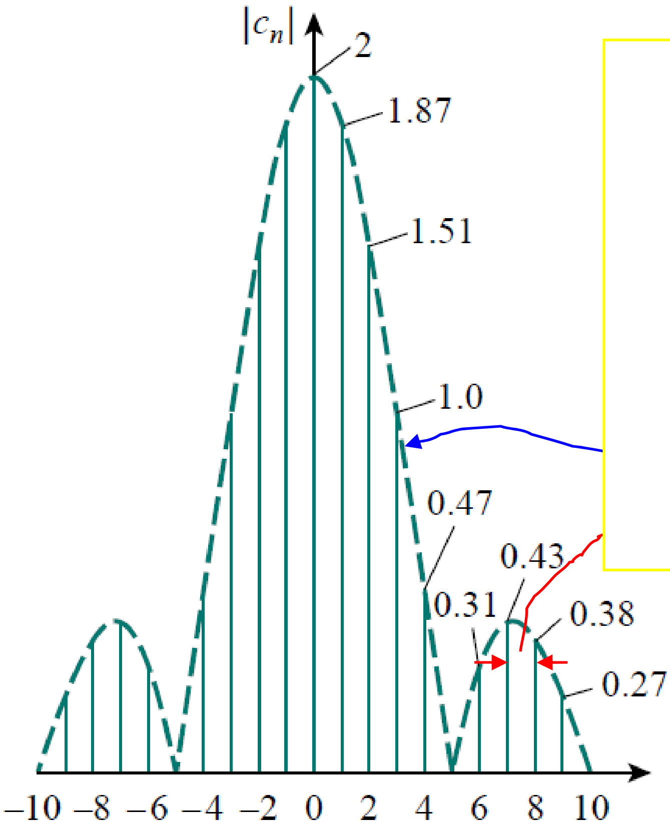
(16.59) =>

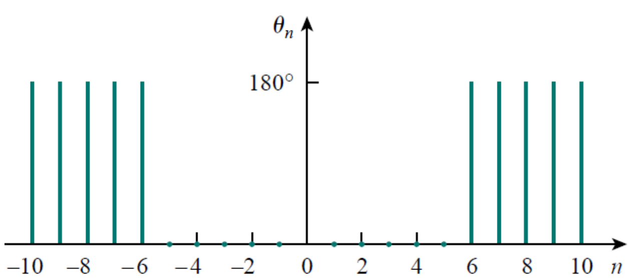
So the complex fourier series is:

Amplitude spectrum:

Phase spectrum:

Plots vs $n = \omega/\omega_0$ (the normalized frequency):





Properties of *sinc* function:

For multiples of π:

Homework: