Chapter 15 – Laplace Transforms (Formulas – seen in Differential Eqs)
Chapter 16 – Laplace Transforms (Circuits)

Chapter 15 –

Motivation:

Sinusoidal → Non-sinusoidal inputs.

Math hard → Use Laplace: Replace T-D diff. eqs: *d/dt* ∫ →
→ F-D algebraic eqs: + − × ÷

Laplace-Transform: analyzes ckts with more than just sinusoidal sources.
It gives the *total response* of a circuit (= "forced" + "natural" responses).

"The Laplace transform is an integral transformation of a function *f*(*t*) from T-D into the complex F-D, giving *F*(*s*)."

or: $f(t) \rightarrow \mathcal{L}[f(t)] = F(s)$

We will transform the T-D ckt $\xrightarrow{\mathcal{L}}$ F-D, solve $\xrightarrow{\mathcal{L}^{-1}\text{(inverse)}}$ T-D.

The Laplace of *f*(t) is:

$$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt \qquad (15.1)$$

where $s = \sigma + j\omega$ (a complex variable (unlike $s=j\omega$ so far)): (15.2)

- $s \cdot t$ is dimensionless => s [Hz or sec^{-1}] units of frequency.
- 0^- denotes "just before $t = 0$" in time, and includes the origin.

To ignore $t < 0$, multiply: $f(t) \rightarrow f(t)\cdot u(t)$ [or write: $f(t),\ t \geq 0$]
T-D function × the unit step function.

Inverse Laplace transform $\mathcal{L}^{-1}$: Is a companion function to $\mathcal{L}$.

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st}\,ds$$

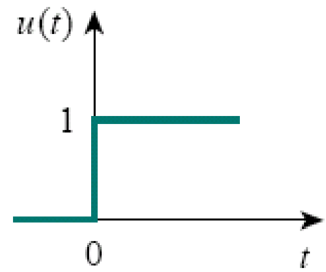
We won't use this – instead → **lookup Table 15.1.**

$f(t) \leftrightarrow F(s)$ are Laplace transform pair:
means 1:1 correspondence between $f(t) \leftrightarrow F(s)$

Example / Laplace transforms of important functions (know these proofs):

$$\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt \qquad (15.1)$$

a) Laplace TF of $u(t)$ – the unit step function:



$$\mathcal{L}[u(t)] = \int_{0^-}^{\infty} 1\cdot e^{-st}dt = -\frac{1}{s}\cdot e^{-st}\Big|_0^{\infty}$$

$$= -\frac{1}{s}\cdot 0 + \frac{1}{s}\cdot 1 = \frac{1}{s}$$

So $\mathcal{L}[u(t)] = 1/s$

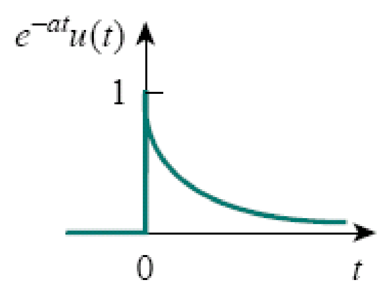
remember:

$\int e^{at}dt = \frac{1}{a}e^{at} + C$, and

$[e^{f(x)}]' = f'(x)\cdot e^{f(x)}$

b) Laplace TF of $e^{-at}u(t)$, $a \geq 0$

$$\mathcal{L}[e^{-at}u(t)] = \int_{0^-}^{\infty} e^{-at}e^{-st}dt$$



$$= -\frac{1}{s+a}\cdot e^{-(s+a)t}\Big|_0^{\infty} = \frac{1}{s+a}$$

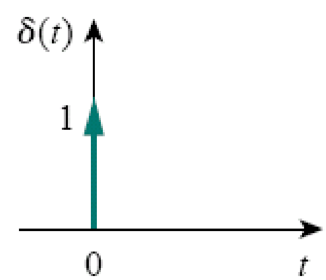
remember: $e^x e^y = e^{(x+y)}$

So $\mathcal{L}[e^{-at}u(t)] = \frac{1}{s+a}$

c) Laplace TF $\delta(t)$ (Dirac delta function)

$\delta(t)$ is defined as: $\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$,

with $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$.



Proof (not in textbook): Integration is made from 0^- to 0^+:

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt = e^{-0} = 1$$

steps:

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{\infty} \delta(t)\cdot e^{-st}dt = \qquad \text{[break limits]}$$

$$= \int_{0^-}^{0^+} \delta(t)\cdot e^{-st}dt + \int_{0^+}^{\infty} \delta(t)\cdot e^{-st}dt = \qquad [\ \delta(t)=0 \text{ fo r} t>0^+\ ]$$

$$= \int_{0^-}^{0^+} \delta(t)e^{-st}dt + \int_{0^+}^{\infty} 0\cdot e^{-st}dt = \qquad [\text{assume } t=0 \text{ for } e^{-st}]$$

$$= \int_{0^-}^{0^+} \delta(t)e^{-s0}dt \qquad [\text{then } e^{-st}=e^{-0}=1)$$

$$= \int_{0^-}^{0^+} \delta(t)\cdot 1\,dt \quad = 1 \qquad [\text{by definition of } \delta(t)]$$

Ex: pp.15.1: Laplace Transform of: $r(t)=tu(t)$ [ramp function]

Sol: $\mathcal{L}[tu(t)] = \int_{0-}^{\infty} te^{-st}dt$

We may integrate by parts: $\int u\,dv = [u\cdot v] - \int v\,du$

let $u=t$	=>	$du = 1\,dt$
let $dv = e^{-st}\,dt$	=>	$v = \frac{-1}{s}\cdot e^{-st}\,dt$

Properties of $\mathcal{L}$:

Linearity

If $F_1(s)$ and $F_2(s)$ are, respectively, the Laplace transforms of $f_1(t)$ and $f_2(t)$, then

$$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s) \quad (15.7)$$

Scaling

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}[f(at)] = \int_0^\infty f(at)e^{-st}\,dt \quad (15.10)$$

, where $a > 0$ is a constant.

=> $$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right) \quad (15.12)$$

Example: Since $$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2} \quad (15.13)$$

$$\mathcal{L}[\sin 2\omega t] = \frac{1}{2}\frac{\omega}{(s/2)^2 + \omega^2} = \frac{2\omega}{s^2 + 4\omega^2} \quad (15.14)$$

which may also be obtained from Eq. (15.13) by replacing ω with 2ω.

Time Shift

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s)$$

Delayed by a in time.

$F(s)$ without the time delay = $\mathcal{L}[f(t)]$, multiplied by e^{-as}.

[AKA time-delay property of Laplace TF.]

Frequency Shift

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}[e^{-at}f(t)] = F(s+a) \quad (15.19)$$

= the Laplace of $f(t)$ by replacing $s \rightarrow s+a$.

[AKA frequency translation.]

Time Differentiation

Given that $F(s)$ is the Laplace transform of $f(t)$

$$\mathcal{L}[f'(t)] = sF(s) - f(0^-) \tag{15.23}$$

For 2nd order derivatives repeat eqn twice or use:

$$\mathcal{L}[f''(t)] = s^2F(s) - sf(0^-) - f'(0^-) \tag{15.24}$$

For higher order → use retrospective Eqn (15.25).

Time Integration

If $F(s)$ is the Laplace transform of $f(t)$

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}F(s) \tag{15.28}$$

Frequency Differentiation

If $F(s)$ is the Laplace transform of $f(t)$,

$$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds} \tag{15.33}$$

Repeated applications of this equation lead to

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^nF(s)}{ds^n} \tag{15.34}$$

Time Periodicity

If function $f(t)$ is a periodic function such as shown in Fig. 15.3, it can be represented as the sum of time-shifted functions shown in Fig. 15.4.

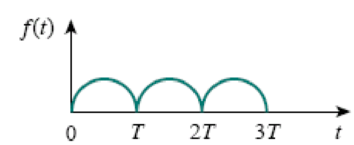


Figure 15.3 A periodic function.

here we obtain: time-shifted Laplace

$$\begin{aligned} F(s) &= F_1(s) + F_1(s)e^{-Ts} + F_1(s)e^{-2Ts} + F_1(s)e^{-3Ts} + \cdots \\ &= F_1(s)[1 + e^{-Ts} + e^{-2Ts} + e^{-3Ts} + \cdots] \end{aligned} \tag{15.38}$$

but

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \tag{15.39}$$

and we end up with:

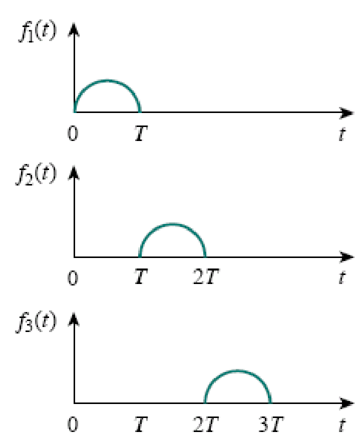


Figure 15.4 Decomposition of the periodic function in Fig. 15.2.

$$F(s) = \frac{F_1(s)}{1 - e^{-Ts}} \qquad (15.40)$$

Laplace of 1st period, only, (divided by $1 - e^{-Ts}$)

Initial and Final Values

We can find $f(0)$ and $f(\infty)$ of $f(t)$ from its Laplace $F(s)$:

$$f(0^+) = \lim_{s\to\infty} sF(s) \qquad (15.42)$$

time-domain *s*-domain AKA initial value theorem

Example: Find $f(0^+)$ if we know the Laplace of $f(t)$:

$$f(t) = e^{-2t}\cos 10t \quad \Longleftrightarrow \quad F(s) = \frac{s+2}{(s+2)^2 + 10^2} \qquad (15.43)$$

Using the initial-value theorem,

$$f(0^+) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{s^2 + 2s}{s^2 + 4s + 104}$$

$$= \lim_{s\to\infty} \frac{1 + 2/s}{1 + 4/s + 104/s^2} = 1$$

[same as replacing t=0 in $f(t)$ => $f(t) = 1\cdot\cos(0) = 1$].

Also:

$$f(\infty) = \lim_{s\to 0} sF(s) \qquad (15.44)$$

AKA final value theorem

For exams: Tables 15.1 and 15.2 on Page 687 will be given.

TABLE 15.1 Properties of the Laplace transform.

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as} F(s)$
Frequency shift	$e^{-at} f(t)$	$F(s+a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2 f}{dt^2}$	$s^2 F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3 f}{dt^3}$	$s^3 F(s) - s^2 f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots - f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(t)\, dt$	$\frac{1}{s} F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds} F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)\, ds$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1 - e^{-sT}}$
Initial value	$f(0^+)$	$\lim_{s \to \infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s \to 0} sF(s)$
Convolution	$f_1(t) * f_1(t)$	$F_1(s) F_2(s)$

TABLE 15.2 Laplace transform pairs.

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t + \theta)$	$\frac{s \sin\theta + \omega \cos\theta}{s^2 + \omega^2}$
$\cos(\omega t + \theta)$	$\frac{s \cos\theta - \omega \sin\theta}{s^2 + \omega^2}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$

[the u(t) is assumed in most time-dependent $f(t)$]

Online: http://www.eecircle.com/applets/007/ILaplace.html

Example 15.3: Find the Laplace TF of: $f(t) = \delta(t) + 2u(t) - 3e^{-2t}\, u(t)$.

Ex. pp.15.3: Find the Laplace TF of: $f(t) = \cos(3t) + e^{-5t}u(t)$.

Ex. pp.15.4: Find the Laplace of: $f(t) = t^2 \cos 3t \, u(t)$.

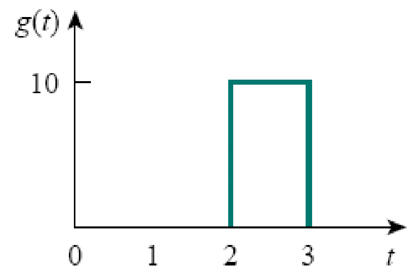
Sol: We know: $\mathcal{L}[\cos 3t] = \frac{s}{s^2+9}$

Notice: $tf(t) \longleftrightarrow -\frac{d}{ds}F(s)$ (Frequency differentiation)

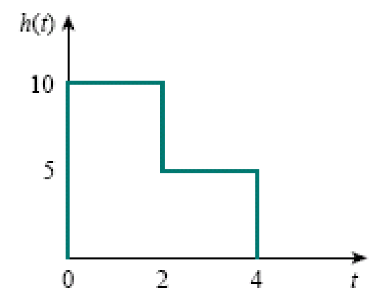
and: $\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}$ (15.34)

thus for n=2, $\mathcal{L}[t^2 \cos 3t] = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+9}\right) =$ (evaluate derivatives)

Ex. 15.5: Find the Laplace transform of the gate function:



Ex. pp. 15.5: Find the Laplace transform of the function:



Ex. 15.6: Find the Laplace transform of the periodic function:

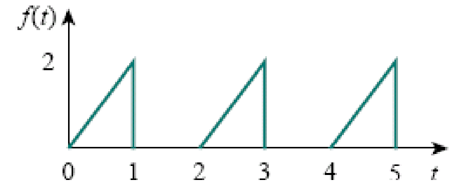


TABLE 15.1 Properties of the Laplace transform.

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Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as} F(s)$
Frequency shift	$e^{-at} f(t)$	$F(s+a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2 f}{dt^2}$	$s^2 F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3 f}{dt^3}$	$s^3 F(s) - s^2 f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots - f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(t)\,dt$	$\frac{1}{s} F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds} F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)\,ds$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
Initial value	$f(0^+)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$
Convolution	$f_1(t) * f_1(t)$	$F_1(s)F_2(s)$

TABLE 15.2 Laplace transform pairs.

$f(t)$	$F(s)$
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t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t + \theta)$	$\frac{s \sin\theta + \omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t + \theta)$	$\frac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Ex. Problem 15.9b: Find the Laplace of: $f(t) = 2e^{-4t}\,u(t-1)$.

TABLE 15.1 Properties of the Laplace transform.

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Scaling	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as}F(s)$
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Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2 f}{dt^2}$	$s^2F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3 f}{dt^3}$	$s^3F(s) - s^2 f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^n f}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots - f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(t)\,dt$	$\frac{1}{s}F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds}F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)\,ds$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
Initial value	$f(0^+)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$
Convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$

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$f(t)$	$F(s)$
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t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t + \theta)$	$\frac{s\sin\theta + \omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t + \theta)$	$\frac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Rules of thumb:

For t in numerator: Frequency differentiation → find Laplace of $f(t)$ & differentiate.

For t in denominator: Frequency integration → find Laplace of $f(t)$ & integrate.

Inverse Laplace Transforms

Given $F(s)$, how do we transform it back to T-D to obtain the corresponding f (t)?

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st}\,ds \qquad (15.5)$$

Avoid integration by matching the entries in Table 15.2

Key: *Partial Fraction Expansion*

If $F(s)$ has the general form of a Transfer Function :

$$F(s) = N(s)\,/\,D(s) \qquad (15.47),$$

we find the inverse Laplace transform $\mathcal{L}^{-1}[F(s)] = f(t)$ in 2 steps:

1. Use *partial fraction expansion* to break $F(s)$ to simple terms whose inverse transforms are in Table 15.2.
2. Find the inverse of each term by matching entries in T.15.2.

Theory and Technique through an example:

[remember: complex poles (solutions) come in conjugate]

Example: Find the inverse Laplace transform of : $F(s) = \dfrac{2s^2 + 3s + 2}{(s+2)(s+1)}$

Solution:

Example 15.11:

Find the inverse Laplace of the F-D function: $H(s) = \dfrac{20}{(s+3)(s^2+8s+25)}$

Sol:

Here $H(s)$ has a pair of complex poles at $s^2 + 8s + 25 = 0$, the $s = -4 \pm j3$.

We do not expand the quadratic and write $H(s)$ as:

$$H(s) = \frac{20}{(s+3)(s^2+8s+25)} = \frac{A}{s+3} + \frac{Bs+C}{(s^2+8s+25)} \qquad (15.11.1)$$

Next, determine the expansion coefficients as shown in textbook

(e.g. by multiplying both sides with $(s+3)(s^2+8s+25)$ =>

$20 = A(s^2 + 8s + 25) + (Bs + C)(s + 3)$ => Separate A, B, C as:

$20 = A(s^2 + 8s + 25) + B(s^2 + 3s) + C(s + 3)$ => Equate the s powers:

s^2: $As^2 + Bs^2 = 0$ => $s^2(A+B) = 0$ => ($s \neq 0$) $A = -B$

s : $8As + 3Bs + Cs = 0$ => $8A + 3B + C = 0$ => $C = -5A$

Constant: $20 = 25A + 3C$ => $\boxed{A=2, \;\; => \;\; B=-2, \quad C=-10.}$

Use that result to help complete the square: [=2ac => determines $c^2=4^2=16$]

$$H(s) = \frac{2}{s+3} - \frac{2s+10}{(s^2+8s+25)} = \frac{2}{s+3} - \frac{2(s+4)+2}{(s+4)^2+9}$$

$$= \frac{2}{s+3} - \frac{2(s+4)}{(s+4)^2+9} - \frac{2}{3}\frac{3}{(s+4)^2+9}$$

[multiply & divide with 3]

Take the inverse of each term:

$$h(t) = 2e^{-3t} - 2e^{-4t}\cos 3t - \frac{2}{3}e^{-4t}\sin 3t$$

or combine $cos3t$ using (9.12): [$A = \sqrt{2^2 + (\frac{2}{3})^2} = 2.108, \quad \theta = \tan^{-1}\frac{\frac{2}{3}}{2} = 18.43^\circ$] => to get

$$=> \; h(t) = 2e^{-3t} - Ae^{-4t}\cos(3t - \theta) \; =>$$

$$h(t) = 2e^{-3t} - 2.108e^{-4t}\cos(3t - 18.43^\circ)$$

Chapter 29

Integration by Partial Fractions

A POLYNOMIAL IN x is a function of the form $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, where the a's are constants, $a_0 \neq 0$, and n is a positive integer including zero.

If two polynomials of the same degree are equal for all values of the variable, the coefficients of the like powers of the variable in the two polynomials are equal.

Every polynomial with real coefficients can be expressed (at least, theoretically) as a product of real linear factors of the form $ax + b$ and real irreducible quadratic factors of the form $ax^2 + bx + c$.

A FUNCTION $F(x) = \dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials, is called a *rational fraction.*

If the degree of $f(x)$ is less than the degree of $g(x)$, $F(x)$ is called *proper*; otherwise, $F(x)$ is called *improper.*

An improper rational fraction can be expressed as the sum of a polynomial and a proper rational fraction. Thus, $\dfrac{x^3}{x^2+1} = x - \dfrac{x}{x^2+1}$.

Every proper rational fraction can be expressed (at least, theoretically) as a sum of simpler fractions (*partial fractions*) whose denominators are of the form $(ax+b)^n$ and $(ax^2+bx+c)^n$, n being a positive integer. Four cases, depending upon the nature of the factors of the denominator, arise.

CASE I. DISTINCT LINEAR FACTORS

To each linear factor $ax + b$ occurring once in the denominator of a proper rational fraction, there corresponds a single partial fraction of the form $\dfrac{A}{ax+b}$, where A is a constant to be determined.

See Problems 1-2.

CASE II. REPEATED LINEAR FACTORS

To each linear factor $ax + b$ occurring n times in the denominator of a proper rational fraction, there corresponds a sum of n partial fractions of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

where the A's are constants to be determined.

See Problems 3-4.

CASE III. DISTINCT QUADRATIC FACTORS

To each irreducible quadratic factor $ax^2 + bx + c$ occurring once in the denominator of a proper rational fraction, there corresponds a single partial fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined.

See Problems 5-6.

CASE IV. REPEATED QUADRATIC FACTORS

To each irreducible quadratic factor ax^2+bx+c occurring n times in the denominator of a proper rational fraction, there corresponds a sum of n partial fractions of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

where the A's and B's are constants to be determined.

See Problems 7-8.

Solved Problems

1. Find $\int \frac{dx}{x^2-4}$.

(a) Factor the denominator: $x^2-4 = (x-2)(x+2)$.

Write $\frac{1}{x^2-4} = \frac{A}{x-2}+\frac{B}{x+2}$ and clear of fractions to obtain

$$(1)\quad 1 = A(x+2)+B(x-2) \quad\text{or}\quad (2)\quad 1 = (A+B)x+(2A-2B)$$

(b) Determine the constants.

General method. Equate coefficients of like powers of x in (2) and solve simultaneously for the constants. Thus, $A+B=0$ and $2A-2B=1$; $A=\frac{1}{4}$ and $B=-\frac{1}{4}$.

Short method. Substitute in (1) the values $x=2$ and $x=-2$ to obtain $1=4A$ and $1=-4B$; then $A=\frac{1}{4}$ and $B=-\frac{1}{4}$, as before. (Note that the values of x used are those for which the denominators of the partial fractions become 0.)

(c) By either method: $\frac{1}{x^2-4} = \frac{\frac{1}{4}}{x-2}-\frac{\frac{1}{4}}{x+2}$ and

$$\int\frac{dx}{x^2-4} = \frac{1}{4}\int\frac{dx}{x-2}-\frac{1}{4}\int\frac{dx}{x+2} = \frac{1}{4}\ln|x-2|-\frac{1}{4}\ln|x+2|+C = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|+C$$

2. Find $\int\frac{(x+1)\,dx}{x^3+x^2-6x}$.

(a) $x^3+x^2-6x = x(x-2)(x+3)$. Then $\frac{x+1}{x^3+x^2-6x} = \frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+3}$ and

$$(1)\quad x+1 = A(x-2)(x+3)+Bx(x+3)+Cx(x-2) \quad\text{or}$$

$$(2)\quad x+1 = (A+B+C)x^2+(A+3B-2C)x-6A$$

(b) *General method.* Solve simultaneously the system of equations

$$A+B+C=0,\quad A+3B-2C=1,\quad\text{and}\quad -6A=1$$

to obtain $A=-1/6$, $B=3/10$, and $C=-2/15$.

Short method. Substitute in (1) the values $x=0$, $x=2$, and $x=-3$ to obtain $1=-6A$ or $A=-1/6$, $3=10B$ or $B=3/10$, and $-2=15C$ or $C=-2/15$.

(c) $$\int\frac{(x+1)\,dx}{x^3+x^2-6x} = -\frac{1}{6}\int\frac{dx}{x}+\frac{3}{10}\int\frac{dx}{x-2}-\frac{2}{15}\int\frac{dx}{x+3}$$

$$= -\frac{1}{6}\ln|x|+\frac{3}{10}\ln|x-2|-\frac{2}{15}\ln|x+3|+C = \ln\frac{|x-2|^{3/10}}{|x|^{1/6}\,|x+3|^{2/15}}+C$$

3. Find $\int\frac{(3x+5)\,dx}{x^3-x^2-x+1}$.

$x^3-x^2-x+1 = (x+1)(x-1)^2$. Then $\frac{3x+5}{x^3-x^2-x+1} = \frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$ and

$$3x+5 = A(x-1)^2+B(x+1)(x-1)+C(x+1)$$

Laplace Transforms with MATLAB

a. Calculate the Laplace Transform using Matlab

Calculating the Laplace F(s) transform of a function f(t) is quite simple in Matlab. First you need to specify that the variable t and s are symbolic ones. This is done with the command

```
>> syms t s
```

Next you define the function f(t). The actual command to calculate the transform is

```
>> F=laplace(f,t,s)
```

To make the expression more readable one can use the commands, `simplify` and `pretty`.

here is an example for the function f(t),

$$f(t) = -1.25 + 3.5te^{-2t} + 1.25e^{-2t}$$

```
>> syms t s
>> f=-1.25+3.5*t*exp(-2*t)+1.25*exp(-2*t);
>> F=laplace(f,t,s)

F =

-5/4/s+7/2/(s+2)^2+5/4/(s+2)

>> simplify(F)

ans =

(s-5)/s/(s+2)^2

>> pretty(ans)
                                    s - 5
                                 ----------
                                           2
                                  s (s + 2)
```

which corresponds to F(s),

$$F(s) = \frac{(s-5)}{s(s+2)^2}$$

Alternatively, one can write the function f(t) directly as part of the laplace command:

```
>>F2=laplace(-1.25+3.5*t*exp(-2*t)+1.25*exp(-2*t))
```

b. <u>Inverse Laplace Transform</u>

The command one uses now is `ilaplace`. One also needs to define the symbols t and s. Lets calculate the inverse of the previous function F(s),

$$F(s) = \frac{(s-5)}{s(s+2)^2}$$

```
>> syms t s
>> F=(s-5)/(s*(s+2)^2);
>> ilaplace(F)
ans =
-5/4+(7/2*t+5/4)*exp(-2*t)
>> simplify(ans)
ans =
-5/4+7/2*t*exp(-2*t)+5/4*exp(-2*t)
>> pretty(ans)
         - 5/4 + 7/2 t exp(-2 t) + 5/4 exp(-2 t)
```

Which corresponds to f(t)

$$f(t) = -1.25 + 3.5te^{-2t} + 1.25e^{-2t}$$

Alternatively one can write

```
>> ilaplace((s-5)/(s*(s+2)^2))
```

Here is another example.

$$F(s) = \frac{10(s+2)}{s(s^2+4s+5)}$$

```
>> F=10*(s+2)/(s*(s^2+4*s+5));
>> ilaplace(F)
ans =
-4*exp(-2*t)*cos(t)+2*exp(-2*t)*sin(t)+4
```

Which gives f(t),

$$f(t) = [4 - 4e^{-2t}\cos(t) + 2e^{-2t}\sin(t)]u(t)$$

making use of the trigonometric relationship,

$$x\sin(\alpha) + y\cos(\alpha) = R\sin(\alpha + \beta)$$

and

$$x\cos(\alpha) - y\sin(\alpha) = R\cos(\alpha + \beta)$$

with

$$R = \sqrt{x^2 + y^2}$$

$$\beta = \tan^{-1}(y/x)$$

One can also write that $f(t) = [4 + 4.47e^{-2t}\cos(t - 153.4^o)]u(t)$

Matlab often gives the inverse Laplace Transform in terms of sinhx and coshx. Using the following definition one can rewrite the hyperbolic expression as a function of exponentials:

$$\sinh(x) = \frac{e^{x} + e^{-x}}{2}$$

$$\cosh(s) = \frac{e^{x} - e^{-x}}{2}$$

Also, you may find the "Heaviside(t) function which corresponds to the unit step function u(t): thus the function H(t) = heaviside(t) =0 for t<0 and H(t) = heaviside(t)=1 for t>0.

As an example, suppose that Matlab gives you the following result for the inverse Laplace transform:

```
2 heaviside(t-10) exp(-5/2t+25) sinh(1/2t-5)
```

This can be re-written, using the definition of the sinh(x) function:

```
2u(t)
```

$$f(t) = 2u(t-10).e^{-2.5(t-10)}[\frac{e^{0.5t-5} - e^{-2.5t+5}}{2}] = u(t-10).e^{-2.5t+25+0.5t-5} - e^{-2.5t+25-0.5t+5}$$

$$= u(t-10)[e^{-2t+20} - e^{-3t+30}]$$

$$= [e^{-2(t-10)} - e^{-3(t-10)}]u(t-10)$$

This last expression is closer to what your hand calculations will give you for the invers Laplace Transform.

Example: Find the Laplace of: $f(t) = 2e^{-4t}\, u(t\text{-}1)$

Solution:

Need to bring to a form in from Table 15.2. Use t-1 for guide.

Multiply and divide by e^{-4} to create the (t-1) in the exponent:

$f(t) = 2\, e^{-4}\, e^{-4(t-1)}\, u(t-1)$ use time-shift with a=1 & $F(s)=L[e^{-4t}u(t)]=1/(s+4)$

$F(s) = 2\, e^{-4}\, e^{-1s}\, (1/(s+4)) =$

$F(s) = 2\, e^{-4}\, e^{-s}\, /(s+4)$.

TABLE 15.1 Properties of the Laplace transform.

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as} F(s)$
Frequency shift	$e^{-at} f(t)$	$F(s+a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2 f}{dt^2}$	$s^2 F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3 f}{dt^3}$	$s^3 F(s) - s^2 f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots - f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(t)\, dt$	$\frac{1}{s} F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds} F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)\, ds$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
Initial value	$f(0^+)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$
Convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$

TABLE 15.2 Laplace transform pairs.

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t + \theta)$	$\frac{s\sin\theta + \omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t + \theta)$	$\frac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

e.g. For t in numerator: Frequency differentiation → find Laplace of $f(t)$ & differentiate.

e.g. For t in denominator: Frequency integration → find Laplace of $f(t)$ & integrate.

15.6 An interesting application: *Integrodifferential Equations*

We can use Laplace to solve equations with derivatives and integrals.

Example:

Use the Laplace transform to solve the differential equation and find $v(t) = ?$:

$$\frac{d^2v(t)}{dt^2} + 6\frac{dv(t)}{dt} + 8v(t) = 2u(t),$$ subject to the conditions: $v(0) = 1$, $v'(0) = -2$

Solution:

Use the time differentiation property:

$\frac{df}{dt}$	$sF(s) - f(0^-)$
$\frac{d^2f}{dt^2}$	$s^2F(s) - sf(0^-) - f'(0^-)$

to take the Laplace of each term and get:

$$[s^2V(s) - sv(0) - v'(0)] \;+\; 6[sV(s) - v(0)] \;+\; 8V(s) \;=\; \frac{2}{s}$$

Substitute the known [$v(0)=1$, $v'(0)=-2$] to get:

$$[s^2V(s) - s + 2\;] + \;6[sV(s) - 1] \;+\; 8V(s) \;=\; \frac{2}{s}$$ => isolate $V(s)$:

$$(s^2 + 6s + 8)V(s) = s + 4 + \frac{2}{s}$$ => solve for $V(s)$

$$V(s) = \frac{s^2 + 4s + 2}{s(s^2 + 6s + 8)}$$ => find poles / simplify

s1,2=[-6 ±√(36-32)]/ 2 = (-2, -4)

$$V(s) = \frac{s^2 + 4s + 2}{s(s+2)(s+4)}$$ => break into fractions

$$V(s) = \frac{s^2 + 4s + 2}{s(s+2)(s+4)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4}$$ => solve as before => … =>

A=1/4, B=1/2, C=1/4

So, $$V(s) = \frac{1/4}{s} + \frac{1/2}{s+2} + \frac{1/4}{s+4}$$ and has inverse Laplace:

$$v(t) = \frac{1}{4}(1 + 2e^{-2t} + e^{-4t})u(t)$$