Time differentiation:

$$\mathcal{L}[f'(t)] = \mathcal{L}\left[\frac{df}{dt}u(t)\right] = \int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\,dt$$

Integrate by parts: $u = e^{-st}$ $du = -se^{-st}\,dt$

$uv - \int v\,du$ $dv = \frac{df}{dt}\,dt$ $v = f(t)$

$$= e^{-st} f(t)\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t)(-s)e^{-st}\,dt$$

$$= \underbrace{e^{-st} f(t)\Big|_{t=\infty}}_{0} - e^{-st} f(t)\Big|_{t=0^-} + s\underbrace{\int_{0^-}^{\infty} f(t)e^{-st}\,dt}_{F(s)}$$

0 f(t) bounded @ ∞

$$\therefore \mathcal{L}[f'(t)] = sF(s) - f(0^-) \qquad (15.23)$$

For 2^{nd}-order differentiation, repeat (15.23) twice:

$$\mathcal{L}[f''(t)] = \mathcal{L}\{[\underbrace{f'(t)}_{=g(t)}]'\} \underset{(15.23)}{=} s\mathcal{L}\{\underbrace{f'(t)}_{g(t)}\} - \underbrace{f'(t)}_{g(t)}\Big|_{t=0^-} = s\mathcal{L}[f'(t)] - f'(0^-)$$

Now, apply (15.23) once again:

$$\mathcal{L}[f''(t)] = s\{sF(s) - f(0^-)\} - f'(0^-) = s^2F(s) - sf(0^-) - f'(0^-) \qquad (15.24)$$

TOPIC:

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FILE UNDER:

Time Periodicity

Consider a function that is periodic with time (begins at time $t=0$ and after that), as illustrated in Fig 15.3. We wish to calculate its Laplace Transform. One way to accomplish this is to regard this periodic function as the infinite summation of individual time shifted functions as in Fig 15.4.

In other words $f(t) = f_1(t) + f_2(t) + f_3(t) + \cdots$ where $f_1(t) = f(t)[u(t) - u(t-T)]$ and is time gated to the interval $0 \to T$ but $f_2, f_3, \ldots$ are all related to f_1 as simply time-shifted functions:

$$f(t) = f_1(t) + f_1(t-T)\,u(t-T) + f_1(t-2T)\,u(t-2T) + \cdots \qquad (15.36)$$

The Laplace transform of (15.36) is the sum of the $\mathcal{L}$ of all the terms.

Using time shift property of the $\mathcal{L}$ (15.17)

$$\mathcal{L}[f(t-a)\,u(t-a)] = e^{-as}F(s)$$

Then,

$$\mathcal{L}[f(t)] = \mathcal{L}[f_1(t)] + \mathcal{L}[f_1(t-T)\,u(t-T)] + \mathcal{L}[f_1(t-2T)\,u(t-2T)] + \cdots$$

Giving $F(s) = F_1(s) + e^{-sT}F_1(s) + e^{-2sT}F_1(s) + \cdots \qquad (15.38)$

$$= F_1(s)\left[1 + e^{-sT} + e^{-2sT} + \cdots\right]$$

But $1 + x + x^2 + x^3 + \cdots = \dfrac{1}{1-x} \qquad (15.39)$

W/ $x = e^{-sT}$ (15.38) becomes

$$F(s) = \frac{F_1(s)}{1 - e^{-Ts}} \qquad (15.40)$$

where $F_1(s)$ is the $\mathcal{L}$ of $f_1(t)$ — over the first period only.

TOPIC:

DATE:

FILE UNDER:

Initial value Theorem

It is possible to determine the initial value of a fct at time $t=0$ $f(t=0)$ directly from its $\mathcal{L}$ $F(s)$. To determine this method, we begin w/ the differentiation property of $\mathcal{L}$.

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-)$$

or using the definition of $\mathcal{L}$:

$$sF(s) - f(0^-) = \int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\, dt \qquad (15.41)$$

Now, if $s \to \infty$, then the integrand in this eqn approaches 0 and this eqn becomes

$$\lim_{s\to\infty} \left[sF(s) - f(0^-)\right] = 0$$

Because $f(0^-)$ is independent of s, we can remove it from the limit, leaving

$$\boxed{f(0^-) = \lim_{s\to\infty} \left[sF(s)\right]} \qquad (15.42)$$

called the Initial-Value Theorem. Neat!

For example, if $f(t) = e^{-2t}\cos(10t)$, then $f(0) = 1 \cdot \cos(0) = \underline{\underline{1}}$

But $F(s) = \mathcal{L}[f(t)] = \mathcal{L}\left[e^{-2t}\cos(10t)\right] =$

$$\mathcal{L}\{\cos(10t)\} = \frac{s}{s^2+10^2}$$

& using the Frequency Shift property $e^{-at}f(t) = F(s+a)$

$$\text{then } F(s) = \left.\frac{s}{s^2+10^2}\right|_{s\to s+2} = \frac{s+2}{(s+2)^2+10^2}$$

Using Initial value theorem

$$f(0^-) = \lim_{s\to\infty}\left[s\cdot\frac{s+2}{(s+2)^2+10^2}\right] \to \lim_{s\to\infty}\left[\frac{s^2}{s^2}\right] = \underline{\underline{1}} \quad \checkmark$$

Final Value Theorem — directly from its Laplace transform.

Similarly, it is possible to determine the "final value" of a time domain fct. That is, it's value as $t \to \infty$.

Beginning w/ (15.41)
$$sF(s) - f(0^-) = \int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\, dt \qquad (15.41)$$

Let's take the limit as $s \to 0$:

$$\lim_{s\to 0}\left[sF(s) - f(0^-)\right] = \lim_{s\to 0}\left[\int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\, dt\right] = \int_{0^-}^{\infty} \frac{df}{dt}\, dt = f(t)\Big|_{0^-}^{\infty}$$

Since $f(0^-) \neq f(s)$ then

$$\lim_{s\to 0}\left[sF(s)\right] - \cancel{f(0^-)} = f(\infty) - \cancel{f(0^-)}$$

or

$$\boxed{f(\infty) = \lim_{s\to 0}\left[sF(s)\right]} \qquad (15.44)$$

This is called the Final Value Theorem.

Once again, consider $f(t) = e^{-2t}\cos(10t)$

$$\lim_{t\to\infty}\{f(t)\} = \lim_{t\to\infty}\left[e^{-2t}\cos(10t)\right] = \underline{\underline{0}}.$$

using the Final value theorem (15.44)

$$f(\infty) = \lim_{s\to 0}\left[sF(s)\right] = \lim_{s\to 0}\left[s \cdot \frac{s+2}{(s+2)^2+10^2}\right]$$

$$= \lim_{s\to 0}\left[\frac{s^2+2s}{s^2+2s+104}\right] \to \lim_{s\to 0} \frac{2s}{2s+104} \to \underline{0} \checkmark$$