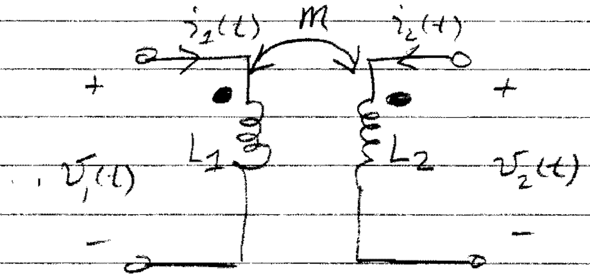
13.3 - Energy in an Inductively Coupled Circuit.

In Ch. 6, you saw previously that the instantaneous energy stored in an inductor is

$$w(t) = \frac{1}{2} L i^2(t) \quad , \text{where } i(-\infty) = 0 \qquad (13.23)$$

Our goal here is to now determine the energy stored in a mutual inductance.

If we consider once again an inductively coupled set of coils as in Fig. 13.14



Assuming the currents are zero initially, so that there is no energy stored, then if we increase i_1 from 0 to I_1 w/ $i_2 = 0$, the instantaneous power in the entire system is

$$p_a(t) = v_1(t) \cdot i_1(t) + \underbrace{v_2(t)}_{\neq 0} \cdot \cancel{i_2(t)}^{0} = i_1 L_1 \frac{di_1}{dt} \qquad (13.24)$$

and the energy stored is

$$w_a = \int_0^t p_a(t)\, dt = \int_0^{t_1} i_1 L_1 \frac{di_1}{\cancel{dt}} \cancel{dt}$$

(13.24)

at $t=0$, $i_1 = 0$; @ t , $i_1(t) = I_1$ $\Longrightarrow$

$$w_a(t) = L_1 \int_0^{I_1} i_1\, di_1 = \frac{1}{2} L_1 i_1^2(t) \Big|_{i_1 = 0\ (t=0)}^{i_1 = I_1\ (t=t)}$$

$$\underline{w_a(t) = \frac{1}{2} L_1 I_1^2} \qquad (13.25)$$

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You've seen such a result before. We'll call that situation a.

Now, let's do something new for you. Let's keep this situation with I_1 in coil 1, but not bring the current in coil 2 from $0 \rightarrow I_2$.

The power in both coils is now

$$P_b(t) = v_1(t) \cdot i_1(t) + v_2(t) \cdot i_2(t)$$

$$= \underbrace{i_1(t) L_1 \cancelto{0}{\frac{di_1(t)}{dt}}}_{i_1 \neq f(t)} + \underbrace{i_1(t)}_{=I_1} \cdot \underbrace{M_{12} \frac{di_2(t)}{dt}}_{(13.16)} + i_2(t) L_2 \frac{di_2(t)}{dt} + i_2(t) \underbrace{M_{21} \cancelto{0}{\frac{di_1(t)}{dt}}}_{(13.11);\ i_1 \neq f(t)}$$

$$= I_1 \cdot M_{12} \frac{di_2(t)}{dt} + i_2(t) L_2 \frac{di_2(t)}{dt} \qquad (13.26)$$

The ~~total~~ energy stored is that of (13.25) plus that due to (13.26):

$$W_b = \tfrac{1}{2} L_1 I_1^2 + \int_0^t I_1 M_{12} \frac{di_2(t)}{dt} dt + \int_0^t i_2(t) L_2 \frac{di_2(t)}{dt} dt$$

$$= \tfrac{1}{2} L_1 I_1^2 + \int_{0\,@\,t=0}^{I_2\,@\,t=t} I_1 M_{12}\, di_2 + \underbrace{\tfrac{1}{2} L_2 I_2}_{\text{as from (13.25)}} \qquad (1)$$

Examining the second term:

$$\int_{0\,@\,t=0}^{I_2\,@\,t=t} I_1 M_{12}\, di_2 = M_{12} I_1 \int_{0\,@\,t=0}^{I_2\,@\,t=t} di_2 = M_{12} I_1 \cdot i_2 \Big|_0^{I_2} = M_{12} I_1 I_2$$

$\therefore$ (1) becomes

$$W_b = \tfrac{1}{2} L_1 I_1^2 + \tfrac{1}{2} L_2 I_2^2 + M_{12} I_1 I_2 \qquad (2)$$

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We see in this result that energy is stored in each inductor, as we would expect, but now there is a third term. Energy is also stored in the mutually inducted field between the two coupled coils.

Finally, let's consider a third scenario, very similar to this last one. But, now let's imagine that port 1 is O.C. s.t. $i_1(t)=0$, and that $i_2(t)$ is brought from 0 to I_2. Then $i_1(t)$ is brought from 0 to I_1. Same final result as scenario b, but just in the opposite order.

If we repeat this derivation for the stored power we would find that

$$W_c = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{21} I_1 I_2 \qquad (13.29)$$

Because the stored energy in the total magnetic field must be the same, then

$$W_b = W_c .$$

For that to be the case, we deduce that $M_{12} = M_{21}$

We'll just define this mutual inductance as M so that

$$M = M_{12} = M_{21} \qquad (13.30a)$$

& from (2) or (13.29)

$$W = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2 \qquad (13.30b)$$

This important equation was derived assuming the currents entered both dotted terminals. If one current enters a dotted terminal & the other leaves a dotted terminal then the mutual voltages will be negative & the mutual energy term will subtract from the total energy as

$$w = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 - M I_1 I_2 \qquad (13.31)$$

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Lastly, since I_1 & I_2 are arbitrary values and we can take the measurement time t to be any time, then I_1 & I_2 can be replaced with i_1 & i_2 and the instantaneous stored energy expression becomes

$$w(t) = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 \pm M i_1 i_2 \qquad (13.32)$$

"+" = both currents enter or both leave the dotted terminals
"−" = one current enters a dotted terminal and the other leaves a dotted terminal.

As shown in your text, there is a limit to the maximum value for M.

$$M \leq \sqrt{L_1 L_2} \qquad (13.35)$$

The coefficient of mutual coupling is defined as

$$k = \frac{M}{\sqrt{L_1 L_2}} \qquad (13.36)$$

And in light of (13.35) cannot be greater than 1 $(0 \leq k \leq 1)$

Referring to Fig 13.15, loosely coupled coils as in Fig 13.15(a) have a smaller k (closer to zero) while strongly coupled coils as in Fig 13.15(b) have a k approaching 1.

$0 \leq k < 0.5$ "loosely" coupled

$0.5 \leq k \leq 1$ "Strongly" coupled

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Example 13.3

For the circuit of Fig 13.16, calculate the coupling coefficient & the energy stored in the coupled inductors at time $t = 1\,s$ if $v = 60\cos(4t + 30°)$ V.

From (13.36) $\quad k = \dfrac{M}{\sqrt{L_1 L_2}} = \dfrac{2.5}{\sqrt{5 \cdot 4}} = \dfrac{2.5}{\sqrt{20}} = 0.56$

(strongly coupled)

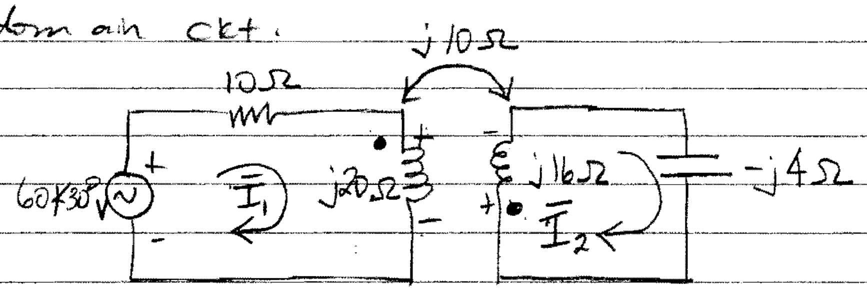
We can use (13.32) to solve for the stored energy at time $t = 1\,s$.

Solve for the currents in the freq. domain. $\omega = 4 \frac{rad}{s}$.

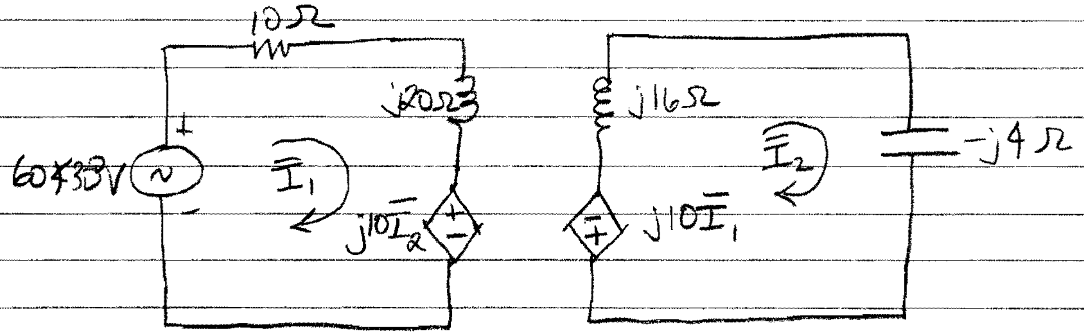
Then, $\quad v = 60\cos(4t + 30°)\,V = 60 \angle 30°\ V$

$$\begin{aligned}
L = 5\,H &\Rightarrow Z_L = j\omega L = j\,4 \cdot 5 = j\,20\,\Omega \\
M = 2.5\,H &\Rightarrow Z_m = j\omega M = j\,4 \cdot 2.5 = j\,10\,\Omega \\
L = 4\,H &\Rightarrow Z_L = j\omega L = j\,4 \cdot 4 = j\,16\,\Omega \\
C = \tfrac{1}{16}\,F &\Rightarrow Z_C = \frac{1}{j\omega C} = \frac{-j}{4/16} = -j\,4\,\Omega
\end{aligned}$$

Frequency domain ckt.



using equivalent ckt model for mutual inductance



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KVL in loop 1: $60 \angle 30^\circ = (10 + j20)\bar{I}_1 + j10\bar{I}_2$ (13.3.1)

KVL in loop 2: $j10\bar{I}_1 + (j16 - j4)\bar{I}_2 = 0$

or $\bar{I}_1 = -\frac{12}{10}\bar{I}_2 = -1.2\bar{I}_2$ (13.3.2)

Sub into (13.3.1). $(10+j20)(-1.2\bar{I}_2) + j10\bar{I}_2 = 60\angle 30^\circ$

or $\bar{I}_2 = 3.254 \angle 160.6^\circ$ A

and $\bar{I}_1 = -1.2\bar{I}_2 = 3.905 \angle -19.40^\circ$ A

Now, in the time domain

$i_1(t) = Re\{\bar{I}_1 \cdot e^{j\omega t}\} = 3.905 \cos(4t - 19.40^\circ)$ A

; $i_2(t) = Re\{\bar{I}_2 \cdot e^{j\omega t}\} = 3.254 \cos(4t + 160.6^\circ)$ A

At time $t = 1$ s, $\omega t = 4t = 4$ rad $= 4 \cdot \frac{180^\circ}{\pi_{rad}} = 229.18^\circ$

$i_1(t=1) = 3.905 \cos(229.2^\circ - 19.4^\circ) = -3.389$ A

$i_2(t=1) = 3.254 \cos(229.2^\circ + 160.6^\circ) = 2.824$ A

∴ The total stored magnetic energy in the coupled inductors is from (13.32)

$w(t=1s) = \frac{1}{2}L_1 i_1^2(t=1) + \frac{1}{2}L_2 i_2^2(t=1) \pm M\, i_1(t=1)\cdot i_2(t=1)$

$= \frac{1}{2} 5 \cdot (-3.389)^2 + \frac{1}{2} 4 (2.824)^2 + 2.5(-3.389)(2.824)$

∴ subtracts.

$\underline{w(t=1s) = 20.74 \text{ J}}$