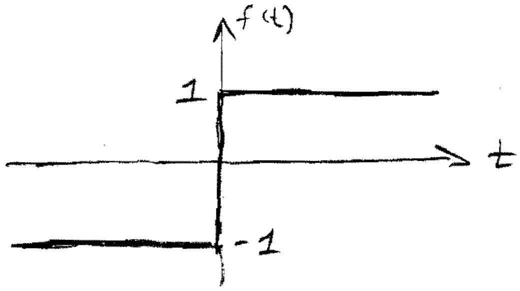
Section 18.3 (cont.)

Example 18.4 Calculate the Fourier transform of the following functions:

(a) signum function $f(t) = \mathrm{sgn}(t)$:



The signum fct can be expressed as

$f(t) = \mathrm{sgn}(t) = u(t) - u(-t)$ (10)

It will be useful to derive the Fourier transform of the unit step function first.

Recall from Chapter 7, eqn. (7.38) $\delta(t) = \dfrac{du(t)}{dt}$

$\therefore \quad du(t) = \delta(t)\,dt$

$\therefore \quad u(t) = \displaystyle\int_{-\infty}^{t} \delta(\tau)\,d\tau$ (11)

From (9), $\mathcal{F}\left\{\displaystyle\int_{-\infty}^{t} f(\tau)\,d\tau\right\} = \dfrac{F(\omega)}{j\omega} + \pi F(0)\,\delta(\omega)$ (12)

In this case, $f(\tau) = \delta(\tau) \Rightarrow F(\omega) = 1$ & $F(0) = 1$

Sub into (11):

$$\mathcal{F}\{u(t)\} = \mathcal{F}\left\{\int_{-\infty}^{t} \delta(\tau)\,d\tau\right\} = \frac{F(\omega)}{j\omega} + \pi F(0)\,\delta(\omega)$$

$$= \frac{1}{j\omega} + \pi\,\delta(\omega)$$ (18.36), (13)

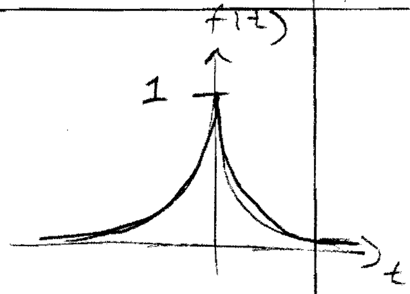
So,

$$\mathcal{F}\{\mathrm{sgn}(t)\} \underset{(10)}{=} \mathcal{F}\{u(t) - u(-t)\} \underset{\text{linearity}}{=} \mathcal{F}\{u(t)\} - \mathcal{F}\{u(-t)\}$$

use (13) for the first terms and use the time reversal property (10) for the second term + (13):

$$\mathcal{F}\{\mathrm{sgn}(t)\} = \frac{1}{j\omega} + \cancel{\pi\delta(\omega)} - \left[\frac{1}{-j\omega} + \cancel{\pi\delta(\omega)}\right] = \frac{2}{j\omega}$$

(b) Double sided exponential $\quad f(t) = e^{-a|t|}$



we can write f(t) as

$$f(t) = \underbrace{e^{-at}u(t)}_{\equiv y(t)} + \underbrace{e^{at}u(-t)}_{= y(-t)}$$

$$Y(\omega) = \mathcal{F}\{y(t)\} = \mathcal{F}\{e^{-at}u(t)\} = \frac{1}{a+j\omega}$$

$$\mathcal{F}\{y(-t)\} = Y(-\omega) = \frac{1}{a-j\omega}$$

$$\therefore \mathcal{F}\{f(t)\} = \mathcal{F}\{y(t)\} + \mathcal{F}\{y(-t)\} = Y(\omega) + Y(-\omega)$$

$$= \frac{1}{a+j\omega} + \frac{1}{a-j\omega} = \frac{1}{a+j\omega}\frac{a-j\omega}{a-j\omega} + \frac{1}{a-j\omega}\cdot\frac{a+j\omega}{a+j\omega}$$

$$= \frac{a-j\omega}{a^2+\omega^2} + \frac{a+j\omega}{a^2+\omega^2} = \underline{\underline{\frac{2a}{a^2+\omega^2}}}$$

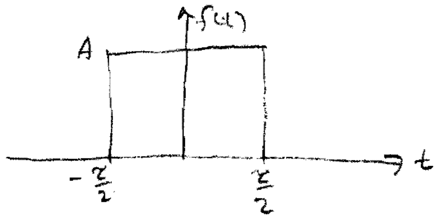
(c) $\text{Sinc}(t) = \dfrac{\sin(t)}{t}$

We'll use the duality property of the Fourier Transform:

$$F(t) \xleftrightarrow{\mathcal{F}} 2\pi f(-\omega)$$

We've seen the sinc function before. In example 18.2 for the pulse

$$f(t) = Au(t+\tfrac{\tau}{2}) - Au(t-\tfrac{\tau}{2}) \xleftrightarrow{\mathcal{F}} A\tau\,\text{sinc}\left(\frac{\omega\tau}{2}\right)$$



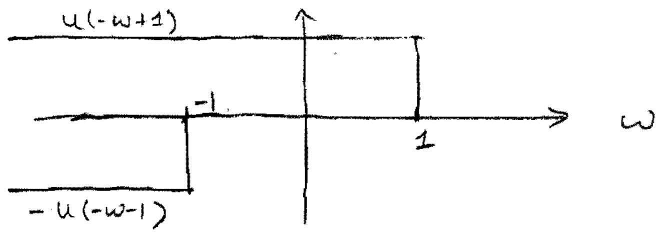
comparing $A\tau\,\text{sinc}\left(\frac{\omega\tau}{2}\right)$ w/ $\text{sinc}(t)$

we'll set $\frac{\tau}{2} = 1$ and $A\tau = 1$. $\left(A\tau\frac{2}{2} = 1 \Rightarrow A = \frac{1}{2}\right)$

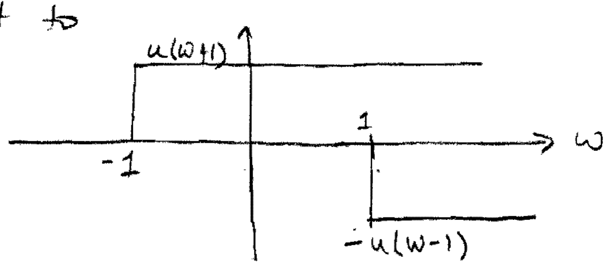
Hence, by the Duality property:

$$\mathcal{F}\{\operatorname{sinc}(t)\} = 2\pi\left[\tfrac{1}{2}u(-\omega+1) - \tfrac{1}{2}u(-\omega-1)\right]$$
$$= \pi\left[u(-\omega+1) - u(-\omega-1)\right]$$

- $u(-\omega+1) = 1$ for $-\omega+1>0$, or $-\omega>-1 \Rightarrow \omega<1$
- $u(-\omega-1) = 1$ for $-\omega-1>0$ or $-\omega>1 \Rightarrow \omega<-1$



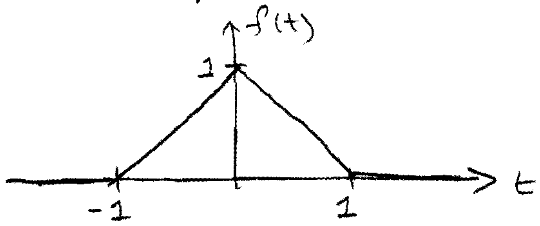
Equivalent to



$\therefore$

$$\underline{\mathcal{F}\{\operatorname{sinc}(t)\} = \pi\left[u(\omega+1) - u(\omega-1)\right]}$$

Example 18.5 Calculate the Fourier transform of $f(t)$ as shown in Fig 18.14.

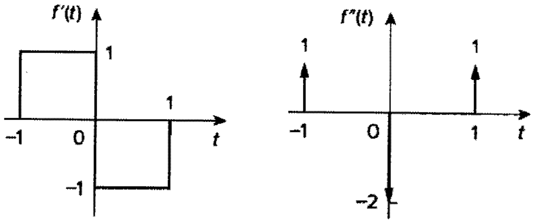


We can calculate this Fourier Transform directly using the definition 18.8:

$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

Instead, we'll illustrate an alternate method that makes this calculation quite easy by taking advantage of the differentiation property of the F.T.

Taking the first derivative in time of $f(t)$ leads to $f'(t)$ in Fig 18.15(a). Then taking the time derivative of this fct leads to the 3 impulses shown in Fig. 18.15(b).



The second derivative can be written as

$$f''(t) = \delta(t+1) - 2\delta(t) + \delta(t-1)$$

Now taking the Fourier transform of this equation gives

$$\underbrace{(j\omega)^2}_{\text{Differentiation}} F(\omega) = \underbrace{1}_{\delta} \cdot \underbrace{e^{+j\omega \cdot 1}}_{\text{time delay}} - 2 + 1 \cdot e^{-j\omega \cdot 1}$$

Now, solve for $F(\omega)$:

$$F(\omega) = \left(\frac{-1}{\omega^2}\right)\left[e^{j\omega} + e^{-j\omega} - 2\right]$$

$$= \frac{1}{\omega^2}\left[2 - 2\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right)\right]$$

$$\therefore \quad F(\omega) = \frac{2[1 - \cos(\omega)]}{\omega^2}$$

To use this approach, keep differentiating until have δ fcts.

Example 18.6 Calculate the inverse Fourier Transform of

(a) $F(\omega) = \dfrac{10j\omega + 4}{(j\omega)^2 + 6j\omega + 8} = \dfrac{10s + 4}{s^2 + 6s + 8}$ this form avoids complex algebra. When finished replace s w/ $j\omega$.

$s = j\omega$

We'll use the same partial fraction expansion process we did in calculating inverse Laplace transform

$$\frac{10s+4}{(s+4)(s+2)} = \frac{A}{s+4} + \frac{B}{s+2} \Rightarrow A = \frac{10(-4)+4}{-2} = 18$$

$$B = \frac{10(-2)+4}{2} = -8$$

$\therefore$

$$F(s) = \frac{18}{s+4} - \frac{8}{s+2}$$

or

$$F(\omega) = \frac{18}{j\omega+4} - \frac{8}{j\omega+2}$$

Using Table 18.2: $\mathcal{F}\left\{\dfrac{1}{a+j\omega}\right\} = e^{-at}u(t)$

then $\underline{f(t)} = \mathcal{F}^{-1}\{F(\omega)\} = \underline{(18e^{-4t} - 8e^{-2t})\,u(t)}$.

(b) $G(\omega) = \dfrac{\omega^2 + 21}{\omega^2 + 9}$ the order of the numerator equals that of the denominator. First perform long division

$$= \frac{\omega^2 + 9 + 12}{\omega^2 + 9} = 1 + \frac{12}{\omega^2 + 9}$$

From Table 18.2 $\mathcal{F}^{-1}\left\{\dfrac{2a}{a^2+\omega^2}\right\} = e^{-a|t|}$

$\therefore$

$$g(t) = \mathcal{F}^{-1}\left\{\frac{\omega^2+21}{\omega^2+9}\right\} = \mathcal{F}\left\{1 + \frac{2\cdot(2\cdot3)}{\omega^2+(3)^2}\right\}$$

$$\underline{g(t) = \delta(\omega) + 2e^{-3|t|}}$$