Section 18.3 - Properties of the Fourier Transform

Similar to Laplace transforms, there are global properties of the Fourier Transform that can greatly aid our calculation of the forward & inverse Fourier transforms.

- **Linearity**: $\mathscr{F}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 F_1(\omega) + a_2 F_2(\omega)$ (18.12), (1)

  Proof is simple & in the text.

- **Time Scaling**: $\mathscr{F}\{f(at)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$ (18.15), (2)

  time expansion ($|a|>1$) leads to frequency compression (spectrum)

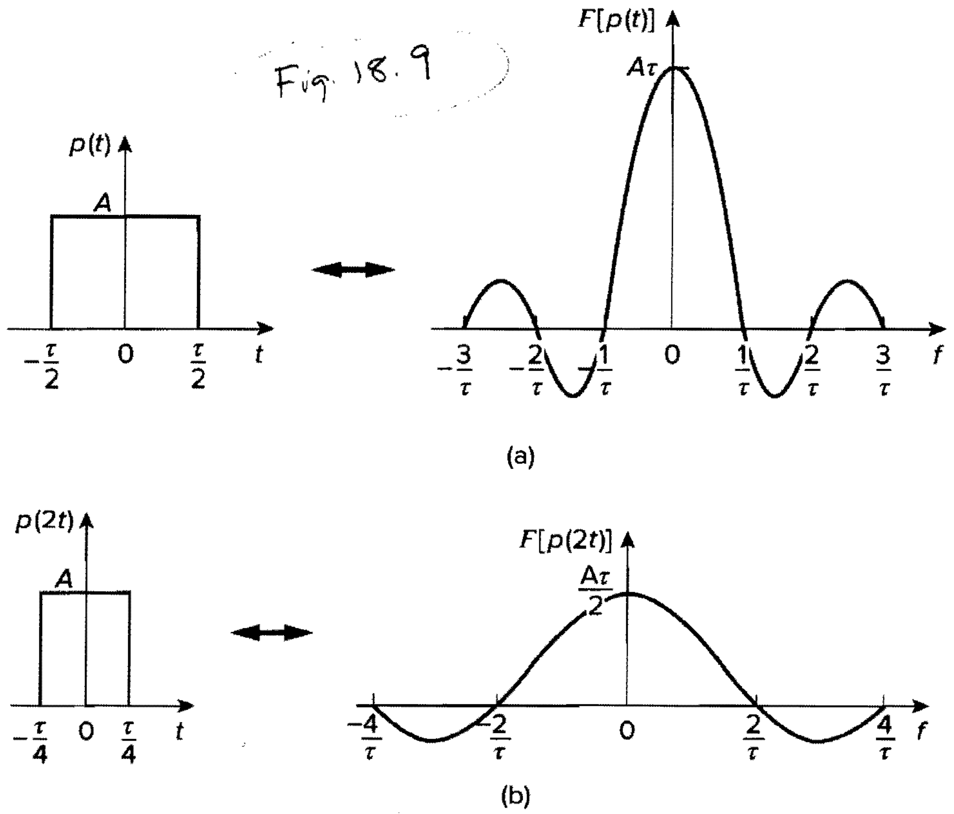
  Proof is simple & in the text.

- **Time Shifting**: $\mathscr{F}\{f(t-t_0)\} = e^{-j\omega t_0} F(\omega)$ (18.20), (3)

  A delay in the time domain leads to Phase shift in frequency domain

  Proof is simple & in the text.

Fig. 18.9



- Frequency Shifting $\mathcal{F}\{f(t)e^{j\omega_0 t}\} = F(\omega-\omega_0)$

This property shows that a frequency shift in the frequency domain corresponds to a phase delay in the time domain.

To prove this, start with the definition

$$\mathcal{F}\{f(t)e^{j\omega_0 t}\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} e^{j\omega_0 t} dt = \underbrace{\int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t} dt}_{= F(\omega-\omega_0)} \qquad (18.25), (4)$$

This last integral is the Fourier transform of $f(t)$, but with $\omega \rightarrow \omega-\omega_0$.

- Modulation: This frequency shifting property is very, very useful in communications.

Modulation is the property of altering the frequency spectrum of a signal in a way that allows for it's efficient transmission from one point to another. There are many types:
- Amplitude modulation (Am)
- Frequency modulation (Fm)
- Phase modulation (Pm)

A simple form of amplitude modulation is to multiply a time domain signal $f(t)$, (which may be voice, music, data) with a so-called carrier tone $\cos(\omega_0 t)$ where ω_0 is most likely a much higher frequency than any component in $f(t)$.

The Fourier transform of this product is

$$\mathcal{F}\{f(t)\cos(\omega_0 t)\} = \mathcal{F}\left\{f(t)\left[\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right]\right\}$$

using linearity property of FT:

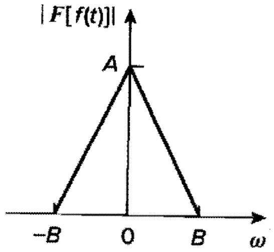
$$= \frac{1}{2}\mathcal{F}\{f(t)e^{j\omega_0 t}\} + \frac{1}{2}\mathcal{F}\{f(t)e^{-j\omega_0 t}\}$$

Applying the Frequency Shifting property in (4) twice gives

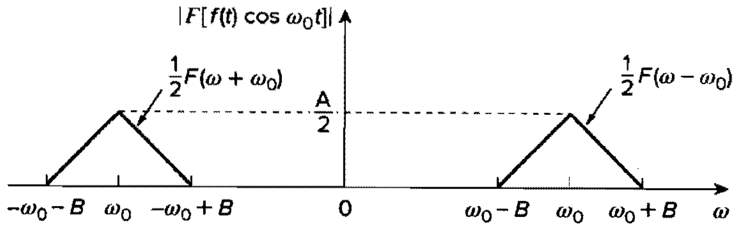
$$\mathcal{F}\{f(t)\cos(\omega_0 t)\} = \frac{1}{2}F(\omega-\omega_0) + \frac{1}{2}F(\omega+\omega_0) \qquad (18.27), (5)$$

If our $f(t)$ had the frequency spectrum of Fig 18.10(a), after multiplying w/ $\cos(\omega_0 t)$ (i.e. modulation), the spectrum would be shifted as shown in Fig 18.10(b).

These higher frequencies might be more efficiently transmitted by an antenna, for example. Cool!!



(a)



(b)

- Time Differentiation : $\mathcal{F}\left\{ \frac{df(t)}{dt} \right\} = j\omega F(\omega)$ (18.28), (6)

To derive this relationship, begin with the definition of $\mathcal{F}^{-1}\{f(t)\}$

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

taking time derivative of both sides

$$\frac{d}{dt} f(t) = \frac{1}{2\pi}\frac{d}{dt}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega \underset{\substack{\text{commute} \\ \text{linear operators}}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, \frac{de^{j\omega t}}{dt}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{j\omega\, F(\omega)}\, e^{j\omega t}\, d\omega$$

This is the Fourier transform of $\frac{df(t)}{dt}$
(There is a significant math error in your text.)

$$\therefore \quad \mathcal{F}\left\{\frac{df(t)}{dt}\right\} = j\omega F(\omega) \qquad (18.30), (7)$$

Repeatedly applying differentiation in this manner yields

$$\mathcal{F}\left\{\frac{d^n f(t)}{dt^n}\right\} = (j\omega)^n F(\omega) \qquad (18.31), (8)$$

- <u>Time integration</u>:

$$\mathcal{F}\left\{\int_{-\infty}^{t} f(\tau)\, d\tau\right\} = \frac{F(\omega)}{j\omega} + \pi F(0)\,\delta(\omega) \qquad (18.34), (9).$$

We'll derive this later once we've discussed convolution.

- <u>Reversal</u>: If $\mathcal{F}\{f(t)\} = F(\omega)$
then $\mathcal{F}\{f(-t)\} = F(-\omega) = F^*(\omega)$ (18.37), (10)

The proof of this property of the Fourier xform is very simple:
It's actually a special case of the Time Scaling property
when $a = -1$:

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

w/ $a = -1$:

$$\mathcal{F}\{f(-t)\} = F(-\omega) \qquad (11)$$

Further, By definition: $F(-\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j(-\omega)t}\, dt = \int_{-\infty}^{\infty} f(t)\, e^{+j\omega t}\, dt$.

Since $f(t)$, t, & ω are all real #'s, then

$$F(-\omega) = F^*(\omega) \qquad (12)$$

- **Duality** – This is a very interesting property of the Fourier Transform, and can be quite useful.

This property states: If $\mathcal{F}\{f(t)\} = F(\omega)$

then

$$\boxed{\mathcal{F}\{F(t)\} = 2\pi f(-\omega)}$$ Weird!

This property is based on the symmetry inherent in the Fourier Transform and the inverse Fourier Transform.

By definition, $f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$ (18.9)

or $$2\pi f(t) = \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega \qquad (13)$$

Now, if we make a simple change of variable

$$t \to -t$$

then (13) becomes $$2\pi f(-t) = \int_{-\infty}^{\infty} F(\omega)\, e^{-j\omega t}\, d\omega \qquad (14)$$

Further, if we interchange variables t & ω in (14), plus $dt \leftrightarrow d\omega$

we obtain $$2\pi f(-\omega) = \underbrace{\int_{-\infty}^{\infty} F(t)\, e^{-j\omega t}\, dt}$$

By definition (18.8), this is the Fourier transform of the function $F(t)$.

$$\therefore \quad 2\pi f(-\omega) = \mathcal{F}\{F(t)\} \qquad (15)$$

called Duality property of the F.T. (8.40),

This seems strange, but can be quite useful. Let's take an example.

Imagine that $f(t) = \delta(t)$.

We have seen that $F(\omega) = 1$

Consequently, by the duality principle,

$$\mathcal{F}\{F(t)\} = 2\pi f(-\omega)$$

then

$$\underline{\mathcal{F}\{1\}} = 2\pi\,\delta(-\omega) = \underline{2\pi\,\delta(\omega)}$$

We must be a little careful w/ this principle. $F(t)$ must be a real number since it's a fct of time.

For example, $f(t) = e^{-at}\,u(t)$.

From Ex. 18.3 $F(\omega) = \frac{1}{a+j\omega}$

By the duality principle

$$\mathcal{F}\{F(t)\} = 2\pi\, f(-\omega)$$

but $F(t) = \frac{1}{a+jt}$

can't have imaginary unit in a time domain fct

TABLE 18.1

Properties of the Fourier transform.

Property	$f(t)$	$F(\omega)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(\omega) + a_2 F_2(\omega)$
Scaling	$f(at)$	$\frac{1}{\|a\|} F\left(\frac{\omega}{a}\right)$
Time shift	$f(t-a)$	$e^{-j\omega a} F(\omega)$
Frequency shift	$e^{j\omega_0 t} f(t)$	$F(\omega - \omega_0)$
Modulation	$\cos(\omega_0 t) f(t)$	$\frac{1}{2}[F(\omega + \omega_0) + F(\omega - \omega_0)]$
Time differentiation	$\frac{df}{dt}$	$j\omega F(\omega)$
	$\frac{d^n f}{dt^n}$	$(j\omega)^n F(\omega)$
Time integration	$\int_{-\infty}^{t} f(t)\,dt$	$\frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$
Frequency differentiation	$t^n f(t)$	$(j)^n \frac{d^n}{d\omega^n} F(\omega)$
Reversal	$f(-t)$	$F(-\omega)$ or $F^*(\omega)$
Duality	$F(t)$	$2\pi f(-\omega)$
Convolution in t	$f_1(t) * f_2(t)$	$F_1(\omega) F_2(\omega)$
Convolution in ω	$f_1(t) f_2(t)$	$\frac{1}{2\pi} F_1(\omega) * F_2(\omega)$