Section 18.2 (cont.) Examples

Example 18.1 Calculate the Fourier Transform of the following three functions:

(a) $\delta(t-t_0)$

$$(18.8), (1)$$

By definition, $F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} \delta(t-t_0)e^{-j\omega t}\,dt$

$$\therefore F(\omega) = e^{-j\omega t_0} \Rightarrow |F(\omega)| = 1 \Leftarrow \text{constant } \forall \omega.$$

$$\measuredangle F(\omega) = -\omega t_0 \text{ rad.}$$

(b) $e^{j\omega_0 t}$ - We'll work this problem in reverse, so to speak. The forward calculation using the definition of $\mathcal{F}\{f(t)\}$ is difficult.

Let's start w/ the Fourier transform $F(\omega) = \delta(\omega - \omega_0)$

Using (18.9) $f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega$ (18.9), (2)

$$\therefore f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}\,d\omega = \frac{1}{2\pi}e^{j\omega_0 t}$$

Consequently $\underline{\underline{e^{j\omega_0 t}}} \overset{\mathcal{F}}{\longleftrightarrow} \underline{\underline{2\pi\,\delta(\omega-\omega_0)}}$

There is a one-to-one correspondence between time domain fcts and their Fourier transforms.

(c) $\cos(\omega_0 t)$

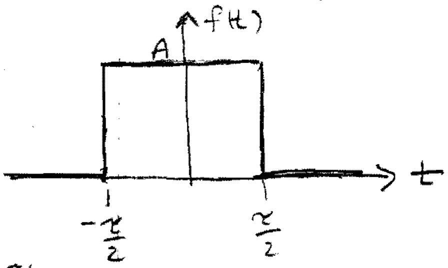
$$\mathcal{F}\{\cos(\omega_0 t)\} = \mathcal{F}\left\{\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right\} = \frac{1}{2}\mathcal{F}\{e^{j\omega_0 t}\} + \frac{1}{2}\mathcal{F}\{e^{-j\omega_0 t}\}$$

$$= \frac{2\pi}{2}\delta(\omega-\omega_0) + \frac{2\pi}{2}\delta(\omega+\omega_0) = \underline{\pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)}$$

(Fig. 18.3 Anagnostou's notes.)

Example 18.2 Derive the Fourier Transform of the pulse function:

Fig 18.4



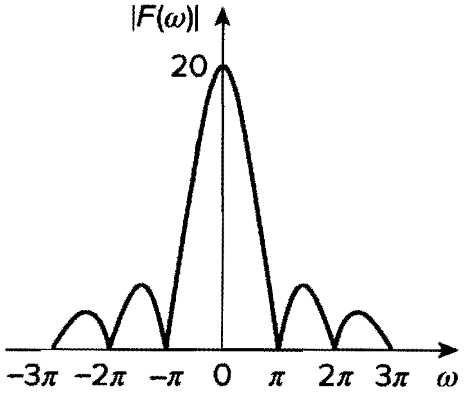
$$\underline{F(\omega)} = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = \int_{-\tau/2}^{\tau/2} A\, e^{-j\omega t}\, dt = \frac{A}{-j\omega}\, e^{-j\omega t}\Big|_{-\tau/2}^{\tau/2} = \frac{A}{-j\omega}\left(e^{-j\omega\frac{\tau}{2}}\right.$$

$$= \frac{A}{-j\omega}\left(e^{-j\omega\frac{\tau}{2}} - e^{+j\omega\frac{\tau}{2}}\right) = \frac{A}{j\omega}\,\frac{\tau}{\tau}\left(e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}\right)$$

$$= A\tau\left(\frac{e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}}{j2\omega\frac{\tau}{2}}\right) = A\tau\,\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}}$$

$$\therefore \quad = \underline{A\tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)}$$

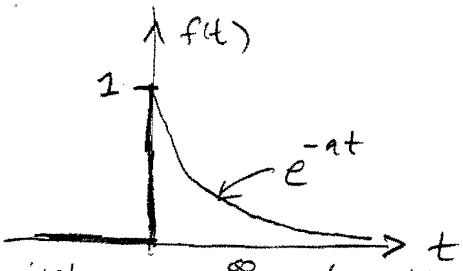
If $A = 10$! $\tau = 2s$, then $F(\omega) = 20\,\mathrm{sinc}(\omega)$



Compare this with the pulse train at the beginning of the Section 18.2 notes. There the spectrum was discrete, w/ a sinc envelope. Here for a single pulse, the spectrum is continuous w/ shape of sinc. Cool!

Example 18.3 Obtain the Fourier x'form of the 'switched-on' decaying exponential

$$f(t) = e^{-at}u(t)$$



$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= -\frac{1}{a+j\omega} e^{-(a+j\omega)t} \Big|_0^{\infty} = -\frac{1}{a+j\omega}(0-1)$$

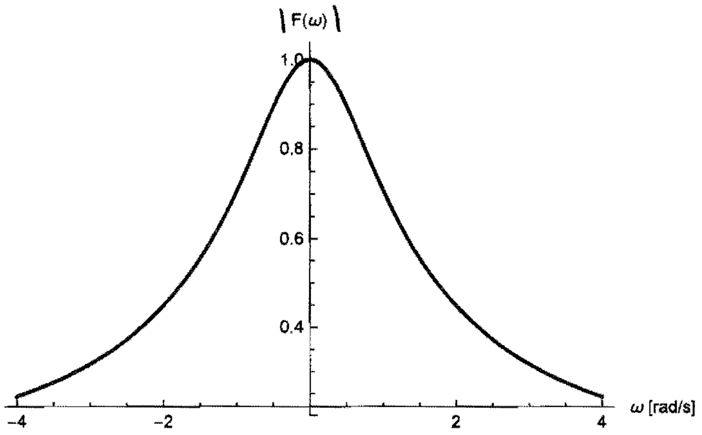
$$\therefore \quad \underline{\underline{F(\omega) = \frac{1}{a+j\omega}}}$$

Whites EE 221 - Circuits II

In[1]:=
```
F[a_, ω_] := 1 / (a + i * ω)
```

In[16]:=
```
a := 1
Plot[Abs[F[a, ω]], {ω, -4 * a, 4 * a}, AxesLabel → {"ω [rad/s]", "F(ω)"}]
```

Out[17]=



In[15]:=
```
Plot[Arg[F[a, ω]] * 180. / Pi, {ω, -4 * a, 4 * a}, AxesLabel → {"ω [rad/s]", "F(ω) [°]"}]
```

Out[15]=

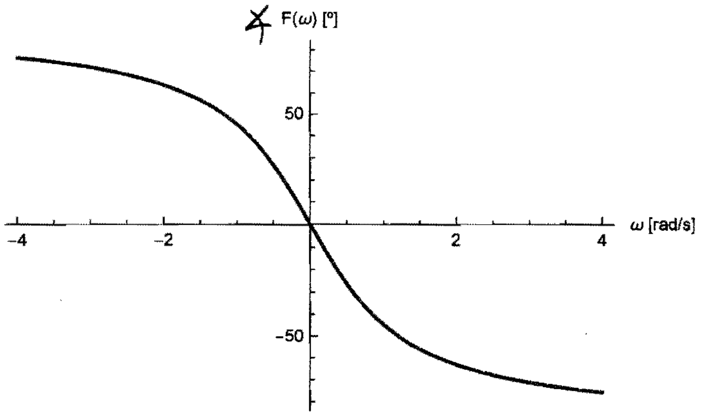


TABLE 18.2

Fourier transform pairs.

$f(t)$	$F(\omega)$
$\delta(t)$	1
1	$2\pi\,\delta(\omega)$
$u(t)$	$\pi\,\delta(\omega) + \dfrac{1}{j\omega}$
$u(t+\tau) - u(t-\tau)$	$2\dfrac{\sin\omega\tau}{\omega}$
$\|t\|$	$\dfrac{-2}{\omega^2}$
$\mathrm{sgn}(t)$	$\dfrac{2}{j\omega}$
$e^{-at}\,u(t)$	$\dfrac{1}{a+j\omega}$
$e^{at}\,u(-t)$	$\dfrac{1}{a-j\omega}$
$t^n e^{-at}\,u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$
$e^{-a\|t\|}$	$\dfrac{2a}{a^2+\omega^2}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$
$\cos\omega_0 t$	$\pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$
$e^{-at}\sin\omega_0 t u(t)$	$\dfrac{\omega_0}{(a+j\omega)^2+\omega_0^2}$
$e^{-at}\cos\omega_0\,t u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$