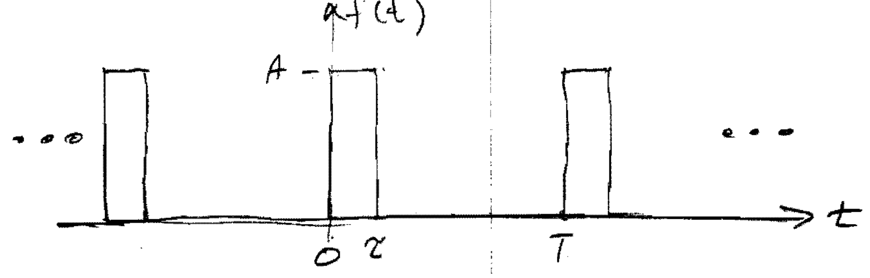
## Section 18.2 Definition of the Fourier Transform

As mentioned in the text, the Fourier Transform allows the transformation of functions to the frequency domain - the calculation of the frequency spectrum - of functions that may not be periodic.

It is informative to consider the frequency spectrum of a periodic function and observe this frequency spectrum becomes very large. This becomes a bridge, so to speak, between periodic and nonperiodic functions as $T \to \infty$ for the periodic function.

Let's quickly reconsider the periodic pulse train example from Section 17.6.




Let's compute the complex Fourier Series for this pulse train.

From (18.1),

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \qquad (1)\ (18.1)$$

Where

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t}\, dt \qquad (2)\ (18.2)$$

For the pulse train above

$$C_n = \frac{1}{T}\int_0^{\tau} A e^{-jn\omega_0 t} = -\frac{A}{jn\omega_0 T} e^{-jn\omega_0 t}\Big|_0^{\tau} = \frac{-A}{jn\omega_0 T}\left(e^{-jn\omega_0 \tau} - 1\right)$$

$$= -\frac{A}{jn\omega_0 T} e^{-jn\omega_0 \frac{\tau}{2}}\left(e^{-jn\omega_0 \frac{\tau}{2}} - e^{jn\omega_0 \frac{\tau}{2}}\right)\cdot\frac{2}{2}$$

$$= +\frac{2A}{n\omega_0 T} e^{-jn\omega_0 \frac{\tau}{2}}\left(\frac{e^{jn\omega_0 \frac{\tau}{2}} - e^{-jn\omega_0 \frac{\tau}{2}}}{2j}\right)$$

$$= \frac{2A}{n\omega_0 T} e^{-jn\omega_0 \frac{\tau}{2}} \sin\left(n\omega_0 \frac{\tau}{2}\right)\cdot\frac{\frac{\tau}{2}}{\frac{\tau}{2}}$$

$$c_n = \frac{2A\frac{\tau}{2}}{T} e^{-jn\omega_0\frac{\tau}{2}} \underbrace{\frac{\sin(n\omega_0\frac{\tau}{2})}{n\omega_0\frac{\tau}{2}}}_{\equiv \operatorname{Sinc}(n\omega_0\frac{\tau}{2})}$$

$$c_n = \frac{A\tau}{T} e^{-jn\omega_0\frac{\tau}{2}} \operatorname{sinc}(n\omega_0\tfrac{\tau}{2})$$

or w/ $\omega_0 = \frac{2\pi}{T}$ , $n\omega_0\frac{\tau}{2} = n\frac{2\pi}{T}\cdot\frac{\tau}{2} = \frac{n\pi\tau}{T}$

$$c_n = \frac{A\tau}{T} e^{-j\frac{n\pi\tau}{T}} \operatorname{sinc}\left(\frac{n\pi\tau}{T}\right) \qquad (3)$$

$\Rightarrow$ $|c_n| = \frac{A\tau}{T}\operatorname{sinc}\left(\frac{n\pi\tau}{T}\right)$

A plot of this spectrum is shown in Fig. 18.2 as $T$ increases, w/ $A = 10$, & $\tau = 0.2$

Notice

- The frequency at which $|c_n| = 0$ remains the same as $T\uparrow$:

  $\operatorname{Sinc}\left(\frac{n\pi\tau}{T}\right) = 0 \Rightarrow \cancel{n\pi}\frac{\tau}{T} = \cancel{n\pi}$

  $\therefore \quad T = \tau \quad$ or $\quad \frac{2\pi}{\omega_0} = \tau \Rightarrow \quad \omega_0 = \frac{2\pi}{\tau}$

  or $f_0 = \frac{1}{\tau} \neq g(n)$

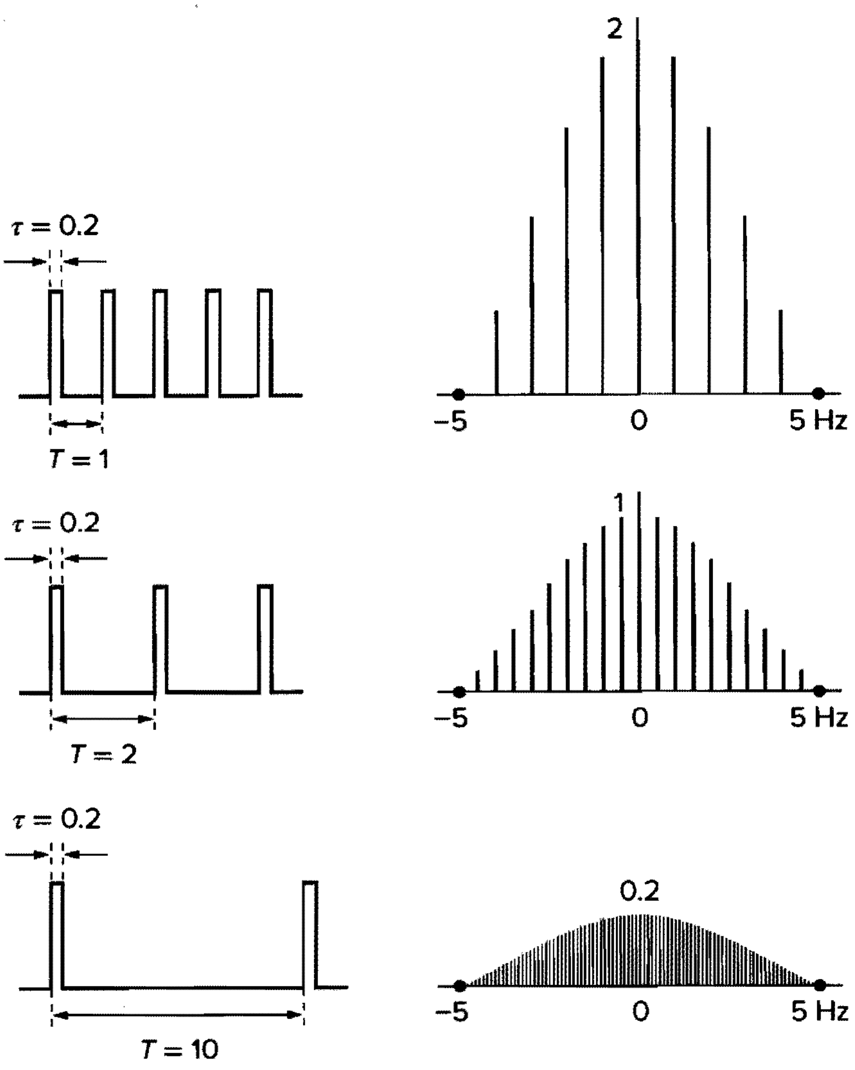
  for $\tau = 0.2 \Rightarrow f_0 = \frac{1}{0.2} = 5$ Hz

- The general shape of the envelope of the spectrum stays the same: $\frac{\sin x}{x}$, as $T\uparrow$

- The # of harmonics between $-f_0$ & $f_0$ $\uparrow$ as $T\uparrow$ (See Fig. 18.2). The spacing between harmonics is decreasing.

- Amplitude of harmonics is decreasing:

$$|c_n| = \underbrace{A\frac{\tau}{T}}_{\downarrow \text{ as } T\uparrow} \operatorname{sinc}\left(\frac{n\pi\tau}{T}\right)$$


τ = 0.2
T = 1
2
−5
0
5 Hz
τ = 0.2
T = 2
1
−5
0
5 Hz
τ = 0.2
T = 10
0.2
−5
0
5 Hz

To further understand this connection of the Fourier series of a periodic function & the Fourier Transform of a non-periodic function, we start w/ (1) & (2)

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \underset{\substack{\uparrow \\ (2)}}{=} \sum_{n=-\infty}^{\infty} \left[\frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt\right] e^{jn\omega_0 t} \quad (4)$$

The spacing between adjacent harmonics is

$$\Delta\omega \equiv (n+1)\omega_0 - n\omega_0 = n\omega_0 - n\omega_0 = \omega_0 = \frac{2\pi}{T}$$

∴ Sub into (4) gives

$$f(t) = \sum_{n=-\infty}^{\infty} \left[\frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt\right] e^{jn\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt\right] \Delta\omega\, e^{jn\omega_0 t} \quad (5)$$

As $T \to \infty$:

- $\sum_{n=-\infty}^{\infty} (\,)\, \Delta\omega\, e^{jn\omega_0 t} \longrightarrow \int_{-\infty}^{\infty} (\,) e^{jn\omega_0 t} d\omega$
- $\Delta\omega \to d\omega$
- $n\omega_0 \to \omega$ Discrete spectrum becomes continuous.

Hence (5) becomes as $T \to \infty$

$$f(t) = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt\right]}_{\equiv \text{ Fourier transform of } f(t)} e^{j\omega t} d\omega}_{\equiv \text{ Inverse Fourier Transform of } F(\omega)} \quad (18.7), (6)$$

By definition, the Fourier Transform of a function $f(t)$ is

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \qquad (18.8), (7)$$

and is an integral transform of $f(t)$ from the time domain to the (real) frequency domain.

$F(\omega)$ is a complex function, in general. The magnitude of $F(\omega)$ is the amplitude spectrum; the angle of $F(\omega)$ is the phase spectrum. Both are, in general, continuous functions of $\omega$.

The inverse Fourier Transform is defined as

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega \qquad (18.9), (8)$$

$f(t)$ & $F(\omega)$ form a one-to-one integral transform pair

$$f(t) \Longleftrightarrow F(\omega) \qquad (18.10), (9)$$

$\mathcal{F}\{f(t)\}$ exists provided the integral in (7) converges.