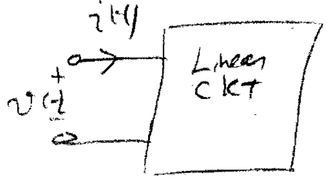
17.5 – Average Power and RMS Values

Let's imagine we have a linear circuit for which the voltage and current have each been expressed in terms of a Fourier Series:



$$v(t) = V_{DC} + \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t - \theta_n) \qquad (1), (17.42)$$

← Amp./phase form

and

$$i(t) = I_{DC} + \sum_{m=1}^{\infty} I_m \cos(m\omega_0 t - \phi_m) \qquad (2), (17.43)$$

← (Different indices, so we don't confuse them later)

We wish now to calculate the time-average power consumed by this passive circuit

$$P = \frac{1}{T}\int_0^T v(t)\cdot i(t)\, dt. \qquad \omega_0 = \frac{2\pi}{T} \qquad (3), (17.44)$$

We only need to integrate over one period of the periodic signals since both have the same fundamental frequency ω_0 because it is a linear circuit.

Sub. (1) & (2) into (3) gives

$$P = \frac{1}{T}\int_0^T V_{DC} I_{DC}\, dt + V_{DC}\cdot\frac{1}{T}\int_0^T \left[\sum_{m=1}^{\infty} I_m \cos(m\omega_0 t - \phi_m)\right] dt$$

$$+ I_{DC}\cdot\frac{1}{T}\int_0^T \left[\sum_{n=1}^{\infty} V_n \cos(n\omega_0 t - \theta_n)\right] dt$$

$$+ \frac{1}{T}\int_0^T \left[\sum_{n=1}^{\infty} V_n \cos(n\omega_0 t - \theta_n)\right]\cdot\left[\sum_{m=1}^{\infty} I_m \cos(m\omega_0 t - \phi_m)\right] dt$$

Recall that integration and summation operators commute meaning we can sum then integrate, or integrate then sum:

$$P = V_{DC} I_{DC} + \sum_{m=1}^{\infty} \frac{I_m V_{DC}}{T}\int_0^T \cos(m\omega_0 t - \phi_m)\, dt$$

$$+ \sum_{n=1}^{\infty} \frac{V_n I_{DC}}{T}\int_0^T \cos(n\omega_0 t - \phi_n)\, dt \qquad (4)$$

$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{V_n I_m}{T}\int_0^T \cos(n\omega_0 t - \theta_n)\cos(m\omega_0 t - \phi_m)\, dt$$

As we saw in Section 17.2 :

$$\left.\begin{aligned}\int_0^T \cos(m\omega_0 t - \phi_m)\,dt &= 0\\ \int_0^T \cos(n\omega_0 t - \phi_n)\,dt &= 0\end{aligned}\right\}$$ Average value of sinusoid = 0.

So the second & third terms in (4) vanish.

The first and last terms, though, generally aren't zero.

For the fourth term, we take advantage of another result from section 17.2 when we were developing the theory of Fourier series. Namely,

$$\int_0^T \cos(n\omega_0 t)\cos(m\omega_0 t)\,dt = 0 \quad m \neq n \qquad (5), (17.4e)$$

Using this orthogonality property of cosine functions, all of the terms in the double summation where $m \neq n$ must equal zero.

Consequently, (4) becomes

$$P = V_{DC} I_{DC} + \frac{1}{T}\sum_{n=1}^{\infty} V_n I_n \int_0^T \cos(n\omega_0 t - \theta_n)\underbrace{\cos(n\omega_0 t - \phi_n)}_{m=n}\,dt \qquad (6)$$

For the remaining terms in the summation,
Applying the trig i.d. $\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$
to the second term gives

$$\cos(\overset{A}{n\omega_0 t - \theta_n})\cos(\overset{B}{n\omega_0 t - \phi_n}) = \frac{1}{2}\cos(n\omega_0 t - \theta_n - n\omega_0 t + \phi_n)$$
$$+ \frac{1}{2}\cos(n\omega_0 t - \theta_n + n\omega_0 t - \phi_n) \qquad (7)$$
$$= \frac{1}{2}\cos(\theta_n - \phi_n) + \frac{1}{2}\cos(2n\omega_0 t - \theta_n - \phi_n)$$

Consequently, sub (7) into the integral of (6) gives

$$\int_0^T \cos(n\omega_0 t - \theta_n)\cos(n\omega_0 t - \phi_n)\,dt =$$
$$\underbrace{\int_0^T \frac{1}{2}\cos(\theta_n - \phi_n)\,dt}_{\frac{T}{2}\cos(\theta_n - \phi_n)} + \underbrace{\int_0^T \frac{1}{2}\cos(2n\omega_0 t - \theta_n - \phi_n)\,dt}_{=0} \qquad (8)$$

The second integral $= 0$ since its time average over 1 period $= 0$.

Sub. (8) into (6) yields

$$P = V_{DC} I_{DC} + \frac{1}{T} \sum_{n=1}^{\infty} V_n I_n \cdot \frac{T}{2} \cos(\theta_n - \phi_n)$$

or

$$P = V_{DC} I_{DC} + \frac{1}{2} \sum_{n=1}^{\infty} V_n I_n \cos(\theta_n - \phi_n) \qquad (9), (17.46)$$

This result shows that the time average power is the DC power plus the sum of the time average powers in each harmonic n. There are no "cross" terms, so to speak. No time average power from a V_n and an I_m, for example. Interesting.

Following a similiar series of steps, it can be shown that the rms value of a periodic function $f(t)$ is given as

$$F_{rms} \equiv \sqrt{\frac{1}{T} \int_0^T f^2(t)\, dt} \qquad \text{w/ } f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n)$$

$$= \sqrt{a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} A_n^2} \qquad (10), (17.49)$$

or

$$= \sqrt{a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)} \qquad \text{w/ } f(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right] \qquad (11), (17.50)$$

In the case that $f(t)$ was a periodic voltage signal & F_{rms} where is rms value, then the time average power dissipated in a resistor R would be

$$P = \frac{F_{rms}^2}{R} \qquad (12), (17.52)$$

whereas if $f(t)$ was a period current signal, the time-average power dissipated in R would be

$$P = F_{rms}^2 \cdot R \qquad (13), (17.51)$$

If $R = 1\,\Omega$, then these two expressions (12) & (13) are identical and apply equally to both voltage & current periodic waveforms.
Hence, w/ $R = 1\,\Omega$, then

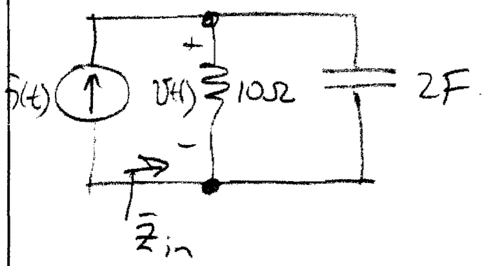
$$P_{1\Omega} = F_{rms}^2 \underset{(11)}{=} a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \qquad (14), (17.51)$$

This is called Parseval's Theorem.

Example 17.8 Calculate the time average power supplied to the circuit below when

$$[t-(-10^\circ)] \qquad [3t-(-35^\circ)]$$

$$i(t) = 2 + 10\cos(t+10^\circ) + 6\cos(3t+35^\circ)\ A$$



Just three terms, rather than an entire infinite summation.

In the frequency domain, $\bar{Z}_{in} = 10 \| \frac{1}{j\omega 2} = \frac{10 \cdot \frac{1}{j\omega 2}}{10 + \frac{1}{j\omega 2}} \cdot \frac{j\omega 2}{j\omega 2} = \frac{10}{1+j20\omega}$

$$\bar{V}(\omega) = \bar{I}(\omega) \cdot \bar{Z}_{in}(\omega) = \frac{10 \cdot \bar{I}}{1+j20\omega} = \frac{10\ \bar{I}}{|1+j20\omega| \angle \tan^{-1}(20\omega)}$$

$$|1+j20\omega| = \sqrt{1^2 + (20\omega)^2} = \sqrt{1+400\omega^2}$$

$$\therefore \quad \bar{V}(\omega) = \frac{10\ \bar{I}}{\sqrt{1+400\omega^2} \angle \tan^{-1}(20\omega)} \qquad (15)$$

We'll use this expression to calculate $\bar{V}$ at ω_0 & $3\omega_0$ for the given $i(t)$.

Solve for I :

- DC The 2F capacitor is an open ckt at DC. Hence w/ $I_{DC} = 2A$ then $\underline{V_{DC} = 2 \cdot 10\Omega = 20V}$. (16)

- AC : $\omega_0 = 1$ rad/s
  - $n = 1$: $\Rightarrow n\omega_0 = 1$ rad/s.

$$i(t)\Big|_{n=1} = 10\cos(t+10^\circ)\ A \Rightarrow \bar{I}(\omega_0) = 10 \angle 10^\circ\ A.$$

Sub into (15) : $\bar{V}(\omega_0) = \dfrac{10 \cdot 10 \angle 10^\circ}{\sqrt{1+400\omega_0^2} \angle \tan^{-1}(20\omega_0)}$

or $\bar{V}(\omega_0) = \dfrac{100}{\sqrt{1+400}} \angle 10^\circ - \tan^{-1}(20 \text{ rad})$ ⊛ !!

$$\therefore \quad \bar{V}(\omega_0) = 4.994 \angle 10 - 87.14^\circ = 4.994\ V \angle -77.14^\circ \qquad (17)$$

- $n=3 \Rightarrow n\omega_0 = 3\cdot 1 = 3$ rad/s.

$$i(t)\Big|_{n=3} = 6\cos(3t+35°)\text{ A} \Rightarrow \bar{I}(3\omega_0) = 6\angle 35°\text{ A}.$$

Sub into (15): $\bar{V}(3\omega_0) = \dfrac{10\cdot 6\angle 35°}{\sqrt{1+400\cdot 3^2}\,\angle\tan^{-1}(20\cdot 3)}$

or

$$\underline{\bar{V}(3\omega_0)} = 0.9999\angle(35° - \tan^{-1}(60\text{ rad}))$$
$$= 1.0\angle(35° - 89.05°)$$
$$= 1.0\angle(-54.05°)\text{ V} \qquad (18)$$

So, in the time domain, the voltage $v(t)$ is found by adding (16) w/ the time domain forms of (17) & (18):

$$v(t) = 20 + \text{Re}\{4.994\angle(-77.14°)\,e^{j1\cdot\omega_0 t}\} + \text{Re}\{1.0\angle(-54.05°)\,e^{j3\cdot\omega_0 t}\}$$

w/ $\omega_0 = 1\,\frac{\text{rad}}{\text{s}}$:

$$v(t) = 20 + 4.994\cos(t - 77.14°) + 1.0\cdot\cos(3t - 54.05°) \qquad (19)$$

So now that we have $v(t)$ & $i(t)$ in a cosine series, as in (1) & (2), let's use (9) to calculate the time-average power supplied by the source.

$$\underline{\underline{P}} = V_{dc} I_{dc} + \frac{1}{2}\sum_{\substack{n=1\\(n\text{ odd})}}^{3} V_n I_n \cos(\theta_n - \phi_n) \qquad (20)$$

! see (2) for def. of θ_n & ϕ_n

$$= 20\cdot 2 + \frac{1}{2}\,4.994\cdot 10\cos[+77.14° - (-10°)]$$
$$+ \frac{1}{2}\,1.0\cdot 6\cos[54.05° - (-35°)]$$
$$= 40 + 24.97\cdot\cos(87.14°) + 3\cos(89.05°)$$
$$= 40 + 1.246 + 0.0497 = \underline{\underline{41.30\text{W}}}$$

very small wrt dc term. Why? V & I are nearly in quadrature. ⇒ very little time average power.
Why nearly in quadrature? $|Z_c| \ll R$.