17.3 Symmetry Considerations

If our periodic function contains certain symmetries, we can take advantage of that to reduce the amount of calculations needed to derive the expansion coeffs (a_n, b_n), and also check our results to see if we've made any obvious errors.

For example, in Example 17.1 in our Fourier series expansion of the pulse waveform, we notice that

1. $a_n = 0 \quad \forall n$
2. $b_n = 0 \quad \forall n \in \{\text{even integers}\}$.

These results "fell out" of the derivation of a_n & b_n.

Actually, though, these two results are the result of the function $f(t)$ having two types of symmetry: Odd symmetry and Half wave symmetry. We'll talk about such symmetry now, beginning w/ even symmetry.

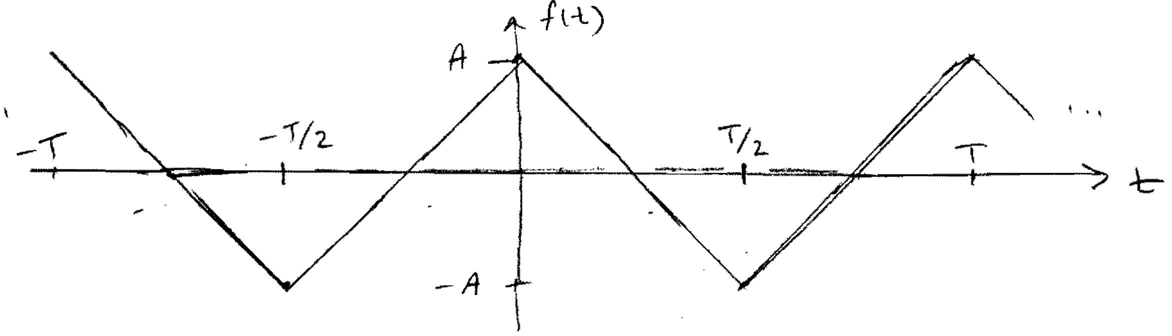
1. Even Symmetry

A function $f(t)$ has even symmetry if

$$f(t) = f(-t) \qquad (1), (17.16)$$

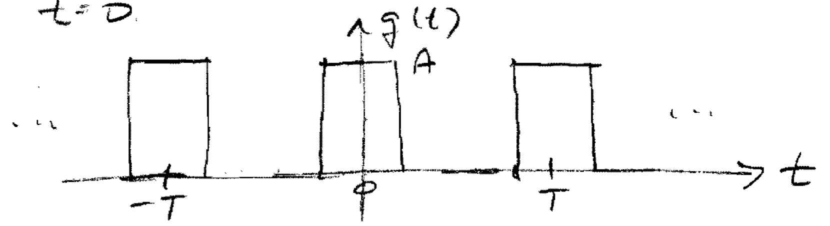
So the symmetry we are interested in is about the vertical axis at time $t=0$. It is a symmetry in time.

An example of an even symmetric function is (Fig. 17.10).



The function "folds" on itself about the vertical $f(t)$ axis at $t=0$.

or



An important property of an even function $f_e(t)$ is that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f_e(t)\, dt = 2\int_0^{T/2} f_e(t)\, dt \qquad (2), (17.17).$$

There are three important outcomes when $f(t)$ has even symmetry ...

1. $$\boxed{a_0 = \frac{2}{T}\int_0^{T/2} f_e(t)\, dt} \qquad (3)$$

To prove this, we begin at the definition of a_0 in (17.6) but over the time period $-T/2$ to $T/2$, which is more convenient time period!

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f_e(t)\, dt \underset{(2)}{=} \frac{2}{T}\int_0^{T/2} f_e(t)\, dt \quad ✓$$

2. $$\boxed{a_n = \frac{4}{T}\int_0^{T/2} f_e(t)\cos(n\omega_0 t)\, dt} \qquad (4)$$

To prove this, we begin w/ the definition of a_n in (17.8) but over the time period $-T/2$ to $T/2$, which is more convenient for this even function $f_e(t)$:

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f_e(t)\cos(n\omega_0 t)\, dt = \frac{2}{T}\int_{-T/2}^{0} f_e(t)\cos(n\omega_0 t)\, dt + \frac{2}{T}\int_0^{T/2} f_e(t)\cos(n\omega_0 t)\, dt$$

In the first integral, we substitute

$$t = -x \Rightarrow dt = -dx \quad \& \quad f_e(t) = f_e(-x) \text{ giving}$$

$$a_n = -\frac{2}{T}\int_{+T/2}^{0} f_e(-x)\cos(-n\omega_0 x)\, dx + \frac{2}{T}\int_0^{T/2} f_e(t)\cos(n\omega_0 t)\, dt$$

$$= \frac{2}{T}\int_0^{T/2} f_e(-x)\cos(n\omega_0 x)\, dx + \frac{2}{T}\int_0^{T/2} f_e(t)\cos(n\omega_0 t)\, dt$$

but for an even function $f(-t) = f(t)$ so these 2 integrals are equal.

Consequently, $a_n = \frac{4}{T}\int_0^{T/2} f_e(t)\cos(n\omega_0 t)\,dt$ ✓

3. $\boxed{b_n = 0 \ \forall n}$ (5)

To prove this, we begin with the definition of b_n in (17.9) but once again over the time period $-T/2$ to $T/2$:

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f_e(t)\sin(n\omega_0 t)\,dt = \frac{2}{T}\int_{-T/2}^{0} f_e(t)\sin(n\omega_0 t)\,dt + \frac{2}{T}\int_0^{T/2} f_e(t)\sin(n\omega_0 t)\,dt$$

Once again, in the first integral we substitute

$$t = -x \Rightarrow dt = -dx \quad \& \quad f_e(-t) = f_e(x) \quad \text{giving}$$

$$b_n = -\frac{2}{T}\int_{+T/2}^{0} f_e(-x)\sin(-n\omega_0 x)\,dx + \frac{2}{T}\int_0^{T/2} f_e(t)\sin(n\omega_0 t)\,dt$$

$$= \frac{2}{T}\int_0^{T/2} f_e(-x)\sin(-n\omega_0 x)\,dx + \frac{2}{T}\int_0^{T/2} f_e(t)\sin(n\omega_0 t)\,dt$$

$$= -\frac{2}{T}\int_0^{T/2} f_e(-x)\sin(n\omega_0 x)\,dx + \frac{2}{T}\int_0^{T/2} f_e(t)\sin(n\omega_0 t)\,dt$$

$$= 0. \quad \text{since} \quad f_e(-t) = f_e(t)$$ ✓

Knowing these properties of even fcts. allows us to more quickly calculate the Fourier Series expansion coeffs. For example, all $b_n = 0$.

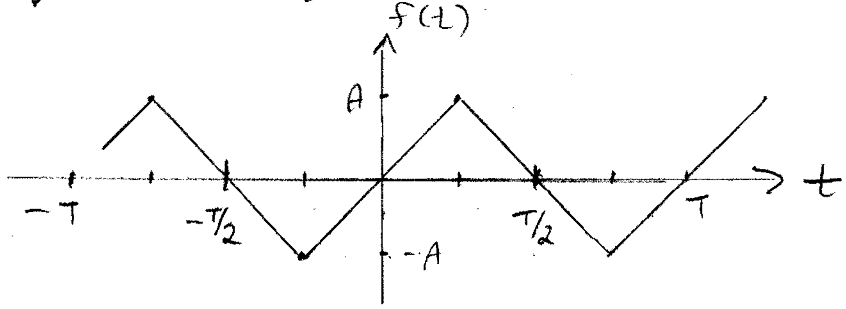
Or if we're given a Fourier series expansion for an even function & $b_n \neq 0 \ \forall n$, then we know there is an error in the expansion.

2. Odd Symmetry

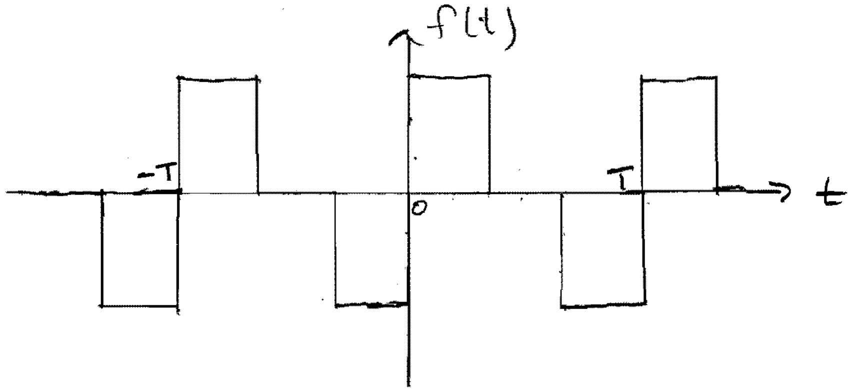
A function $f(t)$ has odd symmetry if

$$f(-t) = -f(t) \qquad (6), (17.26)$$

A couple of examples of odd functions are:



and



If we pivot the waveform for $t > 0$ about the origin, the waveform will overlay with the waveform for $t < 0$. That is odd symmetry in time wrt $t = 0$.

For an odd function of time $f_o(t)$, it can be shown that

$$\boxed{\begin{aligned} a_0 &= 0 \\ a_n &= 0 \quad \forall n \\ \text{\&} \quad b_n &= \frac{4}{T}\int_0^{T/2} f_o(t)\sin(n\omega_0 t)\,dt \end{aligned}} \qquad (7),(17.28)$$

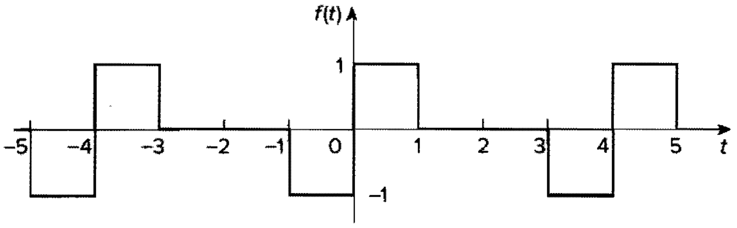
3. Half-Cycle (not wave) Symmetry

~~The third type of symmetry in time that we'll discuss is that for a function that is odd symmetric within a period.~~

~~Some examples include the two functions shown for odd-symmetry above.~~

Example 17.3

Calculate the Fourier series expansion of $f(t)$ shown in Fig 17.13.



This function is odd symmetric about the origin, so from (7).

$$a_0 = 0$$

$$a_n = 0 \qquad \text{\& } T = 4$$

w/ $T/2 = 2$:
$$b_n = \frac{4}{T}\int_0^{T/2} f_o(t)\sin(n\omega_0 t)\,dt$$

$$\therefore\ b_n = \frac{4}{4}\left[\int_0^1 1\cdot\sin(n\omega_0 t)\,dt + \int_1^2 0\cdot\sin(n\omega_0 t)\,dt\right]$$

$$= -\frac{1}{n\omega_0}\cos(n\omega_0 t)\Big|_0^1 = -\frac{1}{n\omega_0}\left[\cos(n\omega_0) - 1\right] = \frac{1}{n\omega_0}\left[1-\cos(n\omega_0)\right]$$

w/
$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$\therefore\ b_n = \frac{2}{\pi n}\cdot\left[1-\cos\left(\frac{n\pi}{2}\right)\right]$$

$$\therefore\ f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1-\cos\left(\frac{n\pi}{2}\right)\right]\cdot\sin\left(\frac{n\pi t}{2}\right)$$

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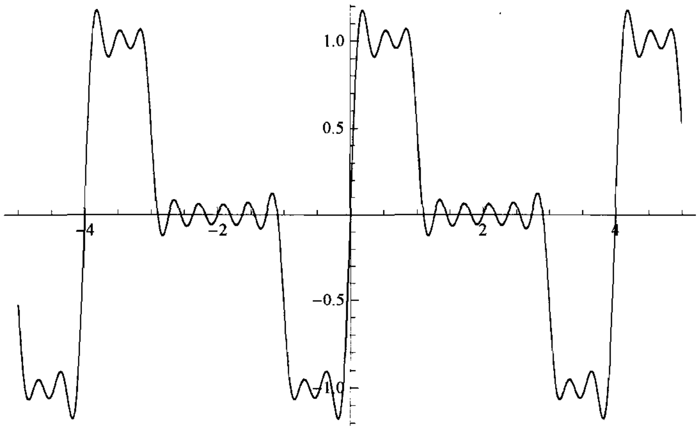
In[14]:=
```
f[nmax_, t_] := 2 / Pi * Sum[1 / n * (1 - Cos[n * Pi / 2]) * Sin[n * Pi * t / 2], {n, nmax}]
b[n_] := 2 / (Pi * n) * (1 - Cos[n * Pi / 2])
```

In[16]:=
```
Plot[f[10, t], {t, -5, 5}]
```

Out[16]=

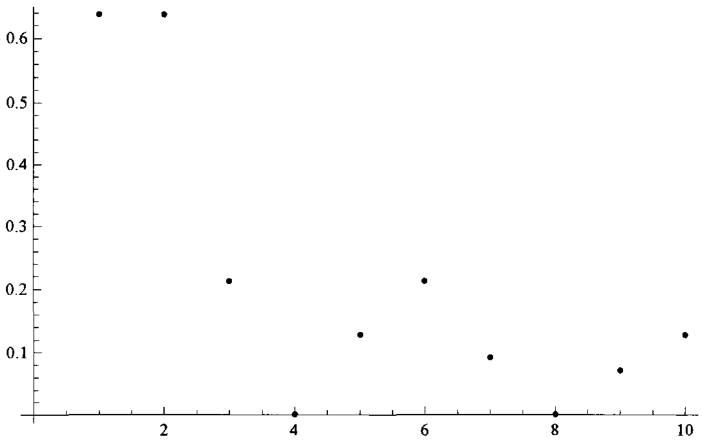


In[17]:=
```
Column[Table[{n, N[b[n]]}, {n, 1, 10}]]
```

Out[17]=
```
{1, 0.63662}
{2, 0.63662}
{3, 0.212207}
{4, 0.}
{5, 0.127324}
{6, 0.212207}
{7, 0.0909457}
{8, 0.}
{9, 0.0707355}
{10, 0.127324}
```

In[18]:=
```
ListPlot[Table[{n, N[b[n]]}, {n, 1, 10}]]
```

Out[18]=



7/7

In[19]:= `Plot[f[50, t], {t, -5, 5}]`

Out[19]=