Chapter 17 - The Fourier Series

We're now going to shift gears here considerably and focus in this chapter on periodic functions as inputs to an electrical circuit. Specifically, as inputs to linear circuits.

Remember that with Laplace transforms we've been working with in the previous two chapters, we could analyze the transient response of linear circuits with quite complicated excitation.

Now, in this chapter, we're going to focus on just periodic excitations and forego any analysis or knowledge of the transient response, or the response to any initial conditions.

This analysis will only provide information on the circuit response to steady-state periodic functions.

Section 17.2 – Trigonometric Fourier Series.

The key tool that we will use in this chapter is something called Fourier's Theorem.

This theorem states that any "practical" periodic function of angular frequency ω_0 can be expressed (or expanded) as an infinite summation of sine & cosine function that are simply integral multiples of ω_0.

That is, according to Fourier's Theorem, $f(t) = f(t+nT)$ (1),(17.1)

can be expressed as

$$f(t) = a_0 + a_1\cos(\omega_0 t) + a_2\cos(2\omega_0 t) + a_3 + \cos(3\omega_0 t) + \cdots + b_1\sin(\omega_0 t) + b_2\sin(2\omega_0 t) + b_3\sin(3\omega_0 t) + \cdots \quad (2),(17.2)$$

or more compactly as

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)\right] \quad (3),(17.3)$$

$\omega_0 = \frac{2\pi}{T}$ is the fundamental angular frequency

So, a_0 represents the time average value of $f(t)$ while the summation in (3) must represent the time varying portion.

$$\left.\begin{array}{l}\cos(n\omega_0 t)\\ \sin(n\omega_0 t)\end{array}\right\} n^{th} \text{ harmonic of } f(t).$$

If n = even, called an even harmonic term
n = odd, called an odd " " .

Amazingly, any well-behaved steady state periodic function can be represented by (3)! What changes between representations of different functions are the coefficients a_n & b_n. That's it!

Simply amazing.

Discovered by Fourier in 1822. He was not believed by the mathematics community. One huge issue was how can a discontinuous function be represented by a summation of only continuous functions? Not obvious.

The Dirichlet conditions were developed & published later giving the conditions upon which a Fourier series representation can exist:

1. $f(t)$ is single valued everywhere
2. $f(t)$ has a finite number of finite discontinuities in one period.
3. $f(t)$ has a finite number of maxima & minima in one period.
4. The integral
$$\int_{t_0}^{t_0+T} |f(t)|\,dt < \infty \quad \text{for any } t_0.$$

Most, or all, of our "practical" inputs & outputs to electrical ckts. will satisfy these conditions.

So, expressing a function in its Fourier series expansion boils down to just calculating a_0, a_n's, & b_n's. That's it.

Sounds simple, and it is, but it's easy to get lost in the details.

Remember, once we've calculated the coefficients a_0, a_n & b_n, we can use (3) as an alternate way of expressing $f(t)$. This is useful, for example, if $f(t)$ is the input voltage or current to a linear circuit. Rather than having to perform some difficult time-domain analysis, we can express the input as in (3) then use a frequency domain phasor analysis for each frequency component $n\omega_0$, summing up all n contributions using superposition to calculate the final output voltage or current, as illustrated in Fig 17.18 & 17.19.

We can fairly easily derive formulas for calculating a_0, a_n & b_n for an arbitrary periodic function $f(t)$.

To begin, we'll calculate a_0. To do this, we'll integrate (3) over one period $t = 0$ to T:

$$\int_0^T f(t)\,dt = \int_0^T a_0\,dt + \int_0^T \left\{ \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right] \right\} dt \qquad (4)$$

We can interchange integration & summation ⇒ i.e., these operators commute!

$$= \int_0^T a_0\,dt + \sum_{n=1}^{\infty} \left[\int_0^T a_n \cos(n\omega_0 t)\,dt + \int_0^T b_n \sin(n\omega_0 t)\,dt \right] \qquad (5)$$

but we know that the average value of cosine & sine functions over 1 period = 0. That is

$$\int_0^T \cos(n\omega_0 t)\,dt = 0 \qquad (6), (17.4b)$$

&

$$\int_0^T \sin(n\omega_0 t)\,dt = 0 \qquad (7), (17.4a)$$

∴ (5) becomes:

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt \qquad (8), (17.6)$$

a_0 is the average value of $f(t)$ over 1 period.

To calculate a_n, we multiply (3) by $\cos(m\omega_0 t)$ & integrate over 1 period: ↑ $m = 1, 2, \ldots$

$$\int_0^T f(t)\cos(m\omega_0 t)\,dt = \int_0^T a_0\cos(m\omega_0 t)\,dt + \int_0^T \left\{\sum_{n=1}^{\infty}\left[a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)\right]\right\}\cos(m\omega_0 t)\,dt$$

We can multiply every term in the summation by $\cos(m\omega_0 t)$ then sum

$$= \int_0^T a_0\cos(m\omega_0 t)\,dt + \int_0^T \left\{\sum_{n=1}^{\infty}\left[a_n\cos(m\omega_0 t)\cos(n\omega_0 t) + b_n\cos(m\omega_0 t)\sin(n\omega_0 t)\right]\right\} dt$$

then interchange order of operators integration & summation:

$$\int_0^T f(t)\cos(m\omega_0 t)\,dt = \int_0^T a_0\cos(m\omega_0 t)\,dt + \sum_{n=1}^{\infty}\left\{\int_0^T a_n\cos(m\omega_0 t)\cos(n\omega_0 t)\,dt + \int_0^T b_n\cos(m\omega_0 t)\sin(n\omega_0 t)\,dt\right\} \qquad (9), (17.7)$$

Now we can use the orthogonality properties of periodic cosine & sine functions:

$$\int_0^T \cos(m\omega_0 t)\sin(n\omega_0 t)\,dt = 0 \qquad (10),\ (17.4c)$$

cosine & sine functions w/ integral multiples of ω_0 are orthogonal over a period.

$$\int_0^T \cos(m\omega_0 t)\cos(n\omega_0 t)\,dt = \begin{cases} 0 & m \neq n \\ \frac{T}{2} & m = n \end{cases} \qquad (11),\ (17.4e,g)$$

cosine functions of different integer frequencies are orthogonal over a period.

and

$$\int_0^T \sin(m\omega_0 t)\sin(n\omega_0 t)\,dt = \begin{cases} 0 & m \neq n \\ \frac{T}{2} & m = n \end{cases} \qquad (12),\ (17.4d,f)$$

sine functions of different integer frequency are orthogonal over one period

Using these orthogonality properties then the terms in the LHS of (9) simplify significantly.

- $$\int_0^T a_0 \cos(m\omega_0 t)\,dt \underset{(6)}{=} a_0 \int_0^T \cos(m\omega_0 t)\,dt = 0. \qquad (13)$$

- $$\int_0^T a_n \cos(m\omega_0 t)\cos(n\omega_0 t)\,dt = a_n \int_0^T \cos(m\omega_0 t)\cos(n\omega_0 t)\,dt = a_n \cdot \Big\{$$

$$\underset{(11)}{=} a_n \cdot \begin{cases} 0 & m \neq n \\ \frac{T}{2} & m = n \end{cases} \qquad (14)$$

and

- $$\int_0^T b_n \cos(m\omega_0 t)\sin(n\omega_0 t)\,dt = b_n \int_0^T \cos(m\omega_0 t)\sin(n\omega_0 t)\,dt$$

$$\underset{(10)}{=} b_n \cdot 0$$

Therefore, (9) simplifies to

$$\int_0^T f(t)\cos(m\omega_0 t)\,dt = a_n \cdot \frac{T}{2}\Big|_{m=n} = a_m \cdot \frac{T}{2}$$

$$\therefore \quad a_m = \frac{2}{T}\int_0^T f(t)\cos(m\omega_0 t)\,dt$$

Replace indice m w/ n since just an arbitrary letter:

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)\,dt \qquad (15),(17.8)$$

We can follow a similiar derivation to determine that

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)\,dt \qquad (16),(17.9)$$

Remember that since the function is periodic, we can extend the limits of integration to an arbitrary time over one period, from t_0 to t_0+T if that makes the calculation of $a_0, a_n, \& b_n$ easier.

Eqns (15) & (16) state that the a_n's are calculated by projecting $f(t)$ onto $\cos(n\omega_0 t)$ over one period, while the b_n's are calculated by projecting $f(t)$ onto $\sin(n\omega_0 t)$.

Example 17.1