Section 16.6.1 Network Stability

The transfer function serves a hugely important role in electrical & computer engineering. We've just seen that for a linear circuit, the unit impulse response $h(t)$ and its Laplace transform $H(s)$ can be used to determine the output for any input. Amazing!

But there's more:

We can also determine the "stability" of a circuit based on its transfer function.

By definition, a circuit is stable if its impulse response $h(t)$ is bounded as $t \to \infty$. This means that $h(t)$ converges to a finite value (zero or otherwise) as $t \to \infty$.

Conversely, a circuit is unstable if $h(t) \to \infty$ as $t \to \infty$.

All passive circuits are stable, as we'll see later in this lecture. Active circuits, though, can be unstable.

The transfer fct $h(t)$ has the Laplace transform $H(s)$. In order for the circuit to be stable, then

$$\lim_{t \to \infty} |h(t)| = \text{finite} \qquad (16.26)$$

Which imposes requirements on $H(s) = \frac{N(s)}{D(s)}$ (16.27)

#1 The first requirement is that the order of $N(s)$ must be less than $D(s)$, i.e., it must be a "proper" function.

To see this, $N(s)$ and $D(s)$ can be expressed as ratios of polynomials in s. By repeated division of the numerator by the denominator would lead to a result

$$H(s) = k_n s^n + k_{n-1} s^{n-1} + \cdots + k_1 s + k_0 + \underbrace{\frac{R(s)}{D(s)}}_{\text{remaining "proper" function}} \qquad (16.29)$$

if the order of the numerator $N(s)$ were larger than the order of the denominator $D(s)$, i.e., an "improper" function.

For example, if $H(s) = \frac{N(s)}{D(s)} = \frac{s^3+4s^2+4s+5}{s^2+3s+2}$

$$= \frac{s^3+3s^2+2s}{s^2+3s+2} + \frac{s^2+2s+5}{s^2+3s+2}$$

$$= s + \frac{s^2+3s+2}{s^2+3s+2} + \frac{-s+3}{s^2+3s+2}$$

$$= \underbrace{s+1}_{\text{improper}} + \underbrace{\frac{-s+3}{s^2+3s+2}}_{\text{proper}} \qquad (1)$$

Now, Apply the final value property of the Laplace X'form

$$f(\infty) = \lim_{s\to\infty}\left[sF(s)\right]$$

to (1)

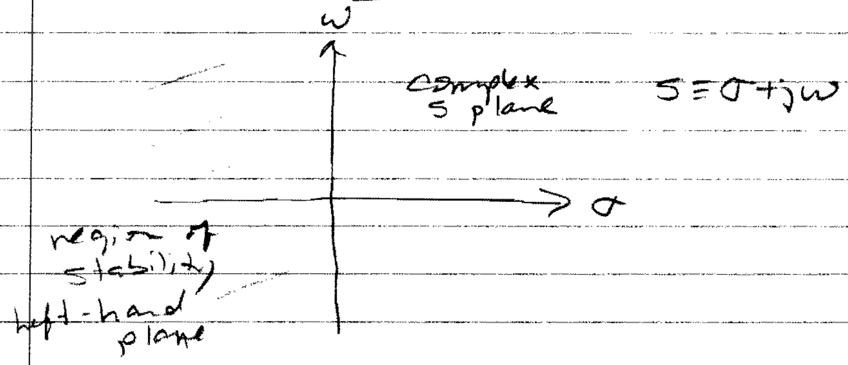
$$f(\infty) = \lim_{s\to\infty}\left[\underbrace{s(s+1)}_{\to\infty} + \underbrace{\frac{s(-s+3)}{s^2+3s+2}}_{\text{finite}}\right]$$

Since $f(\infty) = \infty$, this f.t. is unbounded and the system is unstable

This will be the result for any improper function (16.29) when the order of $H(s)$ is greater than $D(s)$ in (16.27).

#2

The second condition is that all of the poles of $H(s)$ in (16.27) must have negative real parts. Another way of stating this is the poles of $H(s)$ must lie in the left hand plane:



To see this, imagine that a transfer fct. can be expressed with a $D(s)$ that is a product of simple poles.

$$H(s) = \frac{N(s)}{D(s)} = \frac{k N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \qquad (16.28)$$

using a partial fraction expansion w/ residues k_n gives:

$$H(s) = \frac{k_1}{s+p_2} + \frac{k_2}{s+p_2} + \cdots + \frac{k_n}{s+p_n}$$

which has a time domain response

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \left(k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \cdots + k_n e^{-p_n t}\right) u(t) \qquad (16.30)$$

For this result to have a finite value as $t \to \infty$ means than p_n must all have real parts ≥ 0. $\Rightarrow$ $s_i = -p_i$ must lie in the left hand plane (or at most have a zero real part).

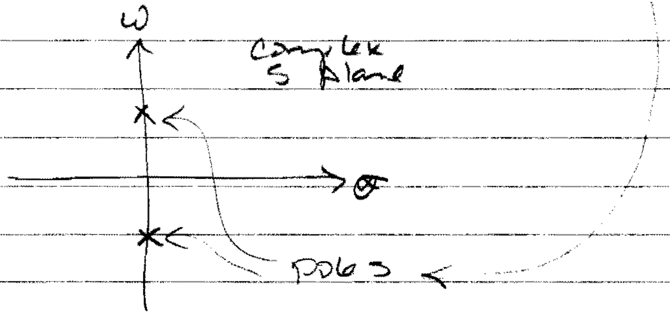
An unstable ckt has poles in the right hand plane which means that $s_i = -p_i$ has a real part < 0 (not '=') meaning that

$$k_i e^{-p_i t} = k_i e^{|p_i| t} \xrightarrow[t \to \infty]{} \infty$$

A ckt comprised of only passive elements (R, L, C) cannot be unstable because this would mean the currents & voltages in the circuit are growing indefinitely with time. This requires signal amplification, which is not available w/ only passive elements.

However, circuits w/ gain such as with transistors or op-amps can be unstable. Conditionally stable circuits are used to make amazing circuits called oscillators. These have poles with a zero real part but w/ an imaginary part $\neq 0$ s.t.

$$H(s) = \frac{N(s)}{s^2 + \omega_0^2} = \frac{N(s)}{(s + j\omega_0)(s - j\omega_0)}$$



The output is a sinusoid

But a circuit comprised of only passive components & dependent sources cannot be unstable. To help see this, let's consider a series RLC ckt in the s domain w/ zero initial conditions.

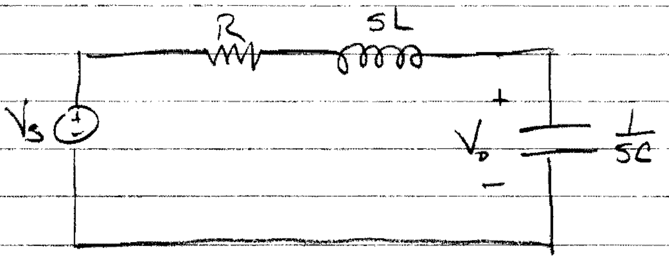


Fig 16.27

By voltage division:

$$H(s) = \frac{V_o}{V_s} = \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} \cdot \frac{sC}{sC} = \frac{1}{s^2LC + sRC + 1} \cdot \frac{\frac{1}{LC}}{\frac{1}{LC}}$$

$$= \frac{\frac{1}{LC}}{s^2 + s\frac{R}{L} + \frac{1}{LC}} \quad \left(= \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} \right) \qquad (16.31)$$

This transfer function has poles at

note: error in text

$$-p_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-R/L \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}$$

$$= -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{\sqrt{LC}}\right)^2}$$

With $\alpha \equiv \frac{R}{2L}$ & $\omega_0 \equiv \frac{1}{\sqrt{LC}}$

$$p_{1,2} = +\alpha \mp \sqrt{\alpha^2 - \omega_0^2} \qquad (16.32)$$

α = attenuation constant (Np)

ω_0 = natural frequency (rad/s)

With $R > 0$, $L > 0$, & $C > 0$:

- if $\alpha^2 > \omega_0^2 \Rightarrow Re\{p_{1,2}\} = +\alpha \mp \sqrt{\alpha^2 - \omega_0^2} \; \underline{\underline{> 0}}$

  $\therefore$ poles $s_{1,2} = -p_{1,2}$ in the LHP $\Rightarrow$ Stable.

- if $\alpha^2 < \omega_0^2 \Rightarrow Re\{p_{1,2}\} = +\alpha > 0$

  $\therefore$ poles $s_{1,2} = -p_{1,2}$ in the LHP.

Example 16.14 An active filter has the Laplace domain transfer fct.

$$H(s) = \frac{k}{s^2 + s(4-k) + 1}.$$

For what values of k is the filter stable?

The poles of the denominator are located at

$$-p_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \underset{\substack{a=c=1 \\ b=4-k}}{\overset{\uparrow}{=}} \frac{-(4-k) \pm \sqrt{(4-k)^2 - 4}}{2}$$

$$= -\tfrac{1}{2}(4-k) \pm \tfrac{1}{2}\sqrt{(4-k)^2 - 4}$$

or $p_{1,2} = \frac{1}{2}(4-k) \mp \frac{1}{2}\sqrt{(4-k)^2 - 4}$

For stable ckt $\Rightarrow$ $Re\{p_{1,2}\} > 0$.

If $4-k > 0$ then first term > 0, second term is real if $(4-k)^2 > 4$, imag. if $(4-k)^2 < 4$.

In either case, $Re\{p_{1,2}\} > 0 \Rightarrow 4-k > 0$

or $\underline{\underline{k < 4}}$