Section 15.3 (cont.)

We will not be using the definition of $\mathcal{L}^{-1}\{\}$ to determine our inverse Laplace transforms. This requires special training in complex plane integration. Fine, though.

Instead, we will use one of two approaches:

1. Directly match entries of table of Laplace x'forms $F(s) \rightarrow f(t)$.
2. Use partial fraction expansion of $F(s) \equiv \frac{N(s)}{D(s)}$, then directly match entries of table of Laplace x'forms.

This will limit what types of functions we can inverse x'form, but they will still be useful for our circuit analysis in Ch. 16.

Method 1 is pretty straight forward.

Example 15.8 Determine the time domain function associated with the Laplace transform

$$F(s) = \frac{3}{s} - \frac{5}{s+1} + \frac{6}{s^2+4}$$

We can use the linearity property of $\mathcal{L}^{-1}$:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{5}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{6}{s^2+4}\right\}$$

Each of these terms has a corresponding entry in Table 5.2.

$$\therefore f(t) = 3u(t) - 5 \cdot e^{-1 \cdot t} u(t) + 3 \cdot \sin(2t)\, u(t)$$

For method 2, our F(s) will be a rational function, a ratio of polynomials. We will use partial fraction expansion to express F(s) in terms that will be identifiable in Table 15.2 so an inverse Laplace transform will be possible.

There are three forms of partial fraction expansions you'll see in this course:

1. Simple poles: In this case, F(s) has the form

$$F(s) = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \tag{15.48}$$

$p_1 \cdots p_n$ are the simple poles, all having the form $(s+p_n)$, and none of the p's are the same.

Note, we're not concerned with the numerator when identifying the poles, other than the degree of N(s) must be < degree of D(s).

Using partial fraction expansion, we can express (15.48) as

$$F(s) = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \cdots + \frac{k_n}{s+p_n} \tag{15.49}$$

where the coeffs. $k_1 \cdots k_n$ are called the residues.

Our quest is to calculate these coeffs. There are a few ways to accomplish this.

The residue method is pretty slick. Multiply both sides of (15.49) by $(s+p_1)$:

$$(s+p_1)F(s) = k_1 + \frac{s+p_1}{s+p_2}k_2 + \cdots + \frac{s+p_1}{s+p_n}k_n \tag{15.50}$$

at $s=-p_1$: $\quad (s+p_1)F(s)\Big|_{s=-p_1} = k_1 \quad$ (15.51)

Very slick! But isn't the LHS = 0? No, there will be a cancellation of $(s+p_1)$ w/ a pole in the denominator of F(s) :)

In general, the residues are calculated by repeating this process for $p_2, \dots, p_n$. A general formula for the residues is then

$$\boxed{k_i = (s+p_i)F(s)\Big|_{s=-p_i}} \qquad (15.52)$$

This is called Heaviside's theorem. The Heaviside "thumb method" really speeds up this residue evaluation process :)

For a Laplace transform of the form (15.49), the associated time domain response using

$$e^{-at} \longleftrightarrow \frac{1}{s+a}$$

is $\quad f(t) = \mathcal{L}^{-1}\{F(s)\} = \left(k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \cdots + k_n e^{-p_n t}\right) u(t) \quad (15.53)$

which is a sum of decaying exponentials.

2. Repeated Simple Poles

This situation is quite similar to case 1 but here there are multiple simple poles at the the same location $-p$:

$$F(s) = \frac{N(s)}{(s+p)^n} + \underbrace{F_1(s)}_{\text{portion of } F(s) \text{ that does not have a repeated pole at } -p.}$$

Then the partial fraction expansion is

$$F(s) = \frac{k_n}{(s+p)^n} + \frac{k_{n-1}}{(s+p)^{n-1}} + \cdots + \frac{k_2}{(s+p)^2} + \frac{k_1}{s+p} + F_1(s) \qquad (15.54)$$

we determine k_n exactly as before by multiplying (15.54) by $(s+p)^n$ and evaluating at $s=-p$:

$$(s+p)^n F(s) = k_n + \frac{(s+p)^n}{(s+p)^{n-1}} k_{n-1} + \cdots + \frac{(s+p)^n}{(s+p)^2} k_2 + \frac{(s+p)^n}{s+p} + (s+p)^n F_1(s)$$

at $s=-p$: $\quad \boxed{k_n = (s+p)^n F(s)\Big|_{s=-p}} \qquad (15.55)$

It's a bit trickier to calculate the other residues. To calculate k_{n-1}, we multiply both sides of (15.54) by $(s+p)^n$

$$(s+p)^n F(s) = k_n + \frac{(s+p)^n}{(s+p)^{n-1}} k_{n-1} + \cdots + \frac{(s+p)^n}{(s+p)^2} k_2 + \frac{(s+p)^n}{s+p} k_1 + (s+p)^n F_1(s)$$

then differentiate wrt s, which has the effect of ridding the k_n term, and evaluating at $s=-p$:

$$\left.\frac{d}{ds}\left[(s+p)^n F(s)\right]\right|_{s=-p} = 0 + \left.\frac{d}{ds}\left[(s+p)k_{n-1} + \cdots + (s+p)^{n-2}k_2 + (s+p)^{n-1}k_1 + (s+p)^n F_1(s)\right]\right|_{s=-p}$$

$$= k_{n-1} \qquad \text{since all other terms} = 0 \text{ at } s=-p.$$

$\therefore$

$$\boxed{k_{n-1} = \left.\frac{d}{ds}\left[(s+p)^n F(s)\right]\right|_{s=-p}} \qquad (15.56)$$

Continuing on with this process, can show that

$$\boxed{k_{n-2} = \left.\frac{1}{2!}\frac{d^2}{ds^2}\left[(s+p)^n F(s)\right]\right|_{s=-p}} \qquad (15.57)$$

while in general,

$$\boxed{k_{n-m} = \left.\frac{1}{m!}\frac{d^m}{ds^m}\left[(s+p)^n F(s)\right]\right|_{s=-p}} \qquad (15.58)$$

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Example 15.9

Find $f(t)$ given $F(s) = \dfrac{s^2+12}{s(s+2)(s+3)}$

Here we have three distinct simple poles. Use a partial fraction expansion:

$$F(s) = \frac{s^2+12}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$$

Use the Heaviside "thumb" method to evaluate the residues:

$$A = \left.\frac{s^2+12}{(s+2)(s+3)}\right|_{s=0} = \frac{12}{6} = 2$$

$$B = \left.\frac{s^2+12}{s(s+3)}\right|_{s=-2} = \frac{4+12}{-2(1)} = -8$$

$$C = \left.\frac{s^2+12}{s(s+2)}\right|_{s=-3} = \frac{9+12}{-3(-1)} = 7$$

$$\therefore \quad F(s) = \frac{2}{s} - \frac{8}{s+2} + \frac{7}{s+3}$$

Using Table 15.2

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \underline{2u(t) - 8e^{-2t}u(t) + 7e^{-3(t)}u(t)}$$

Example 15.10

Calculate $v(t)$ given $V(s) = \dfrac{10s^2+4}{s(s+1)(s+2)^2}$

This Xfrm has simple poles and 1 repeated pole. use a partial fraction expansion.

$$V(s) = \frac{10s^2+4}{s(s+1)(s+2)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+2)^2} + \frac{D}{s+2}$$

Notice!!

we can evaluate A, B, & C use our Heaviside thm's method:

$$A = \left.\frac{10s^2+4}{(s+1)(s+2)^2}\right|_{s=0} = \frac{4}{1\cdot 4} = 1$$

$$B = \left.\frac{10s^2+4}{s(s+2)^2}\right|_{s=-1} = \frac{10+4}{(-1)(1)^2} = -14$$

$$C = \left.\frac{10s^2+4}{s(s+1)}\right|_{s=-2} = \frac{10\cdot 4+4}{(-2)(-1)} = +22$$

To evaluate the residue for the repeated pole at -2, i.e., the coeff D, use (15.56)

$$D = \frac{d}{ds}\left[(s+p)^n F(s)\right]_{s=p} = \frac{d}{ds}\left[(s+2)^2 \cdot V(s)\right]\Big|_{s=-2}$$

$$= \frac{d}{ds}\left[\cancel{(s+2)^2}\cdot \frac{10s^2+4}{s(s+1)\cancel{(s+2)^2}}\right]\Big|_{s=-2} = \frac{d}{ds}\left[\frac{10s^2+4}{s^2+s}\right]\Big|_{s=-2}$$

$$= \left.\frac{(s^2+s)(20s)-(10s^2+4)(2s+1)}{(s^2+s)^2}\right|_{s=-2}$$

$$= \frac{(4-2)(-40)-(10\cdot 4+4)(-4+1)}{(4-2)^2} = \frac{-80+132}{4} = 13$$

$\therefore$

$$V(s) = \frac{1}{s} - \frac{14}{s+1} + \frac{13}{s+2} + \frac{22}{(s+2)^2}$$

use Table 15.2:

$$v(t) = u(t) - 14e^{-t}u(t) + 13e^{-2t}u(t) + 22te^{-2t}u(t)$$

3. Complex Conjugate Poles

(Why? →)

We will consider only a simple (i.e., non-repeated) complex pole. These always occur in complex conjugate pairs since N(s) & D(s) must have real coefficients.

A general form for this type of pole is

$$F(s) = \frac{A_1 s + A_2}{s^2 + as + b} + F_1(s) \qquad (15.61)$$

where $F_1(s)$ is the additional part of $F(s)$ that does not contain this type of pole. (Maybe it's a simple pole or repeated simple pole.)

We could separate the denominator as

$$\begin{aligned}(s^2 + as + b) &= (s + \alpha + j\beta)(s + \alpha - j\beta) \\ &= s^2 + \alpha s \cancel{- j\beta s} + \alpha s + \alpha^2 \cancel{- j\alpha\beta} \\ &\quad \cancel{+ j\beta s} \cancel{+ j\alpha\beta} + \beta^2 \\ &= s^2 + 2\alpha s + \alpha^2 + \beta^2\end{aligned}$$

and use the simple, non-repeated pole approach.

Your text advocates completing the square in the denominator of (15.61) because this former approach involves complex #'s. This latter approach does not.

To complete the square in the denominator of (15.61) we wish to rewrite it in the form

$$(s+\alpha)^2 + \beta^2$$

the sum of two squares.

This allows us to use one of the x'form pairs:

$$\mathcal{L}\{e^{-at}\sin(\omega t)\} \longleftrightarrow \frac{\omega}{(s+a)^2 + \omega^2}$$

01 $$\mathcal{L}\{e^{-at}\cos(\omega t)\} \longleftrightarrow \frac{s+a}{(s+a)^2 + \omega^2}$$

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Example 15.11 Determine the time domain function $h(t)$ associated w/ the Laplace transform

$$H(s) = \frac{20}{(s+3)(s^2+8s+25)}$$

Will use partial fraction expansion to express $H(s)$ as

$$H(s) = \frac{20}{(s+3)(s^2+8s+25)} \underset{(15.61)}{=} \frac{A}{s+3} + \frac{Bs+C}{s^2+8s+25} \qquad (1)$$

using Heaviside thumb rule;

$$\underline{A} = \left.\frac{20}{s^2+8s+25}\right|_{s=-3} = \frac{20}{9-24+25} = \underline{2}$$

There is no "thumb" rule to determine B & C instead we use an algebraic approach. Multiplying both sides of (1) by the denominator of $H(s) = (s+3)(s^2+8s+25)$ yields

$$20 = A(s^2+8s+25) + (Bs+C)(s+3)$$

w/ A=2:

$$20 = 2s^2 + 16s + 50 + Bs^2 + 3Bs + Cs + 3C$$

$$= (B+2)s^2 + (3B+C+16)s + 3C + 50.$$

Equate terms on the LHS & RHS w/ the same power of s:

s^0 : $20 = 3C + 50 \Rightarrow 3C = -30$ or $\underline{C = -10}$

s^2 : $B + 2 = 0$ or $\underline{B = -2}$

s^1 : not needed since we already solved for A.

$$\therefore\ H(s) = \frac{2}{s+3} - \frac{2s+10}{s^2+8s+25}$$

We now wish to determine $h(t)$. Here is where we need to complete the square in the denominator of $H(s)$ and express it in the form listed on p. 7.

Proceeding, $H(s) = \frac{2}{s+3} - \frac{2s+10}{(s+4)^2+9} = \frac{2}{s-3} - \frac{2(s+4)+2}{(s+4)^2+9}$

Looking at Table 15.2, we see in the last term something of the form

$$\frac{s+a}{(s+a)^2+\omega^2}$$

So let's write that term as

$$H(s) = \frac{2}{s+3} - 2\,\frac{s+4}{(s+4)^2+9} - \frac{2}{(s+4)^2+9}$$

Now, we can see something in this last term of the form

$$\frac{\omega}{(s+a)^2+\omega^2}$$

So, let's rewrite that term as

$$H(s) = \frac{2}{s-3} - 2\,\frac{s+4}{(s+4)^2+9} - \frac{2}{3}\,\frac{3}{(s+4)^2+9}$$

From Table 15.2, we can now form the inverse Laplace transform h(t) as

$$h(t) = 2e^{-3t}u(t) - 2e^{-4t}\cos(3t)\,u(t) - \frac{2}{3}e^{-4t}\sin(3t)\,u(t)$$

Quite an interesting time domain waveform. Damped sinusoids.