Prob. 12.1

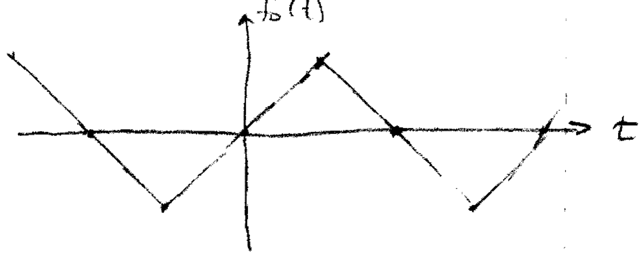
Derive (17.28) using a high degree of detail.

For the trigonometric form of the Fourier series

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right] \qquad (17.3)$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)\,dt \qquad (17.9), (1)$$

For an odd function in t about the $t=0$ axis $f_o(t)$, for example the function



$$f(-t) = -f(t) \qquad (17.26), (2)$$

We can express (1) as

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_o(t)\sin(n\omega_0 t)\,dt = \frac{2}{T}\int_{-\frac{T}{2}}^{0} f_o(t)\sin(n\omega_0 t)\,dt + \frac{2}{T}\int_0^{T/2} f_o(t)\sin(n\omega_0 t)\,dt$$

We may the substitution $t = -t \Rightarrow dt = -dt$ in the first integral in the RHS:

$$b_n = -\frac{2}{T}\int_{+\frac{T}{2}}^{0} f_o(-t)\sin(-n\omega_0 t)\,dt + \frac{2}{T}\int_0^{T/2} f_o(t)\sin(n\omega_0 t)\,dt$$

$$= -\frac{2}{T}\int_0^{T/2} f_o(-t)\sin(n\omega_0 t)\,dt + \frac{2}{T}\int_0^{T/2} f_o(t)\sin(n\omega_0 t)\,dt$$

Sub. (2) into the first integral yields

$$b_n = \frac{2}{T}\int_0^{T/2} f_o(t)\sin(n\omega_0 t)\,dt + \frac{2}{T}\int_0^{T/2} f_o(t)\sin(n\omega_0 t)\,dt$$

$$\therefore \quad \boxed{b_n = \frac{4}{T}\int_0^{T/2} f_o(t)\sin(n\omega_0 t)\,dt}$$

We can express (17.6) as

$$a_0 = \int_{-\frac{T}{2}}^{T/2} f_0(t)\,dt = \int_{-T/2}^{0} f_0(t)\,dt + \int_{0}^{T/2} f_0(t)\,dt$$

substituting $t = -t \Rightarrow dt = -dt$ in the first integral yields

$$a_0 = -\int_{+\frac{T}{2}}^{0} f_0(-t)\,dt + \int_{0}^{T/2} f_0(t)\,dt = \int_{0}^{T/2} f_0(-t)\,dt + \int_{0}^{T/2} f_0(t)\,dt$$

substituting (2) into the first integral gives

$$\underline{\underline{a_0}} = -\int_{0}^{T/2} f_0(t)\,dt + \int_{0}^{T/2} f_0(t)\,dt = \underline{\underline{0}}$$

We can express (17.8) as

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f_0(t)\cos(n\omega_0 t)\,dt = \frac{2}{T}\int_{-\frac{T}{2}}^{0} f_0(t)\cos(n\omega_0 t)\,dt + \int_{0} f_0(t)\cos(n\omega_0 t)\,dt$$

substituting $t = -t \Rightarrow dt = -dt$ in the first integral gives

$$a_n = -\frac{2}{T}\int_{+T/2}^{0} f_0(-t)\cos(-n\omega_0 t)\,dt + \int_{0}^{T/2} f_0(t)\cos(n\omega_0 t)\,dt$$

$$= \frac{2}{T}\int_{0}^{T/2} f_0(-t)\cos(n\omega_0 t)\,dt + \int_{0}^{T/2} f_0(t)\cos(n\omega_0 t)\,dt\,.$$

substituting (2) into the first integral gives

$$a_n = -\frac{2}{T}\int_{0}^{T/2} f_0(t)\cos(n\omega_0 t)\,dt + \int_{0}^{T/2} f_0(t)\cos(n\omega_0 t)\,dt$$

$$\underline{\underline{a_n = 0}}$$

Q.E.D.

P.P.17.4 f(t) = 32t/π, 0 < t < π, T = 2π, ω_o = 1 $= \frac{2\pi}{T}$

This is an even function, b_n = 0. ← (17.18)

(17.18): $$a_o = \frac{2}{T}\int_0^{T/2} f_e(t)dt = \frac{2}{2\pi}\left[\int_0^{\pi} (32t/\pi)dt\right] = \frac{1}{\pi^2}\frac{32t^2}{2}\Big|_0^{\pi} = \mathbf{16}$$

(17.18): $$a_n = \frac{4}{T}\int_0^{T/2} f_e(t)\cos(n\omega_o t)dt = \frac{4}{2\pi}\left[\int_0^{\pi}\frac{32t}{\pi}\cos ntdt\right]$$

$u = 32t \Rightarrow du = 32$

$dv = \cos(nt)dt \Rightarrow v = \frac{1}{n}\sin(nt)$

$$= uv\Big|_0^{\pi} - \int_0^{\pi} v\,du$$

$$= \frac{2}{\pi^2}\left[\underbrace{[(32t/n)\sin nt]_0^{\pi}}_{0} - \frac{32}{n}\int_0^{\pi}\sin ntdt\right]$$

$$= \frac{(-2)}{n\pi^2}\frac{(-32)}{n}\cos nt\Big|_0^{\pi} = \frac{64}{n^2\pi^2}(\underbrace{\cos n\pi}_{=(-1)^n} - 1)$$

= **−128/($n^2\pi^2$), n = odd**
0, n = even

$$f(t) = 16 - \frac{128}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{n^2}\cos nt,$$ **n = 2k − 1**

or $$f(t) = 16 - \frac{128}{\pi^2}\sum_{\substack{n=1\\(n\ odd)}}^{\infty}\frac{\cos(nt)}{n^2}$$

Solution 17.14

Find the quadrature (cosine and sine) form of the Fourier series

$$f(t) = 7.5 + \sum_{n=1}^{\infty} \frac{37.5}{n^3+1} cos\left(\underbrace{2nt}_{B} + \underbrace{\frac{n\pi}{4}}_{A}\right)$$

Solution

Since cos(A + B) = cos A cos B – sin A sin B.

$$\text{f(t)} = 7.5 + \sum_{n=1}^{\infty}\left(\underbrace{\frac{37.5}{n^3+1}\cos(n\pi/4)}_{a_n}\cos(2nt) - \underbrace{\frac{37.5}{n^3+1}\sin(n\pi/4)}_{b_n}\sin(2nt)\right)$$

$$f(t) = 10 + \sum_{n=1}^{\infty}\left[\underbrace{\frac{4}{n^2+1}}_{D}\cos(10nt) + \underbrace{\frac{1}{n^3}}_{E}\sin(10nt)\right]$$

Solution 17.15

(a) $D\cos\omega t + E\sin\omega t = A\cos(\omega t + \theta)$ ← We wish to express in this form

$\therefore$ $A = \sqrt{D^2 + E^2}$, $\theta = -\tan^{-1}(E/D)$

$$A = \sqrt{\frac{16}{(n^2+1)^2} + \frac{1}{n^6}}, \quad \theta = -\tan^{-1}((n^2+1)/(4n^3))$$

$$f(t) = 10 + \sum_{n=1}^{\infty}\sqrt{\frac{16}{(n^2+1)^2} + \frac{1}{n^6}}\cos\left(10nt - \tan^{-1}\frac{n^2+1}{4n^3}\right)$$

in phasor form $D + Ee^{-j\frac{\pi}{2}} = A\angle\theta$

$D - jE = A\angle\theta$

$A = |D - jE| = \sqrt{D^2 + E^2}$

& $\theta = -\tan^{-1}\left(\frac{E}{D}\right)$

Solution 17.10

Find the exponential Fourier series for the waveform in Fig. 17.53.

(17.58) $f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$

where

(17.59) $C_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t}\, dt$

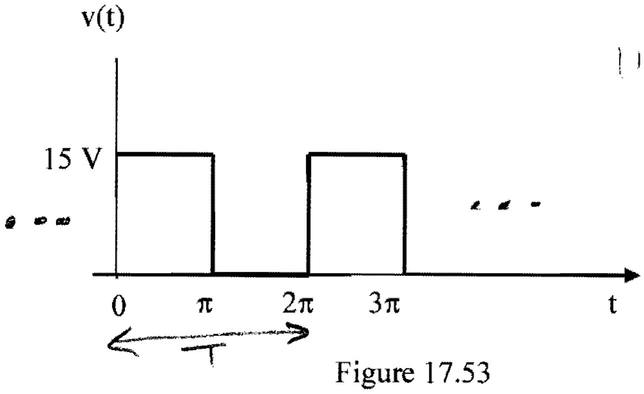


Figure 17.53
For Prob. 17.10.

Solution

$T = 2\pi, \quad \omega_o = 2\pi / T = 1$

$C_0 = \frac{1}{T}\int_0^T f(t)\,dt = 7.5$

(17.59): $c_n = \frac{1}{T}\int_0^T f(t) e^{-jn\omega_o t} dt = \frac{15}{2\pi}\int_0^{\pi} e^{-jnt} dt = \frac{7.5}{\pi}\frac{e^{-jnt}}{(-jn)}\Big|_0^{\pi}$ for $n \neq 0$

$= [7.5/(n\pi)][je^{-jn\pi} - j] = [j7.5/(n\pi)][\cos(n\pi) - 1]$

$e^{-jn\pi} = \cos(n\pi) - j\cancel{\sin(n\pi)}^{\,0}$

$\therefore f(t) = 7.5 + \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{j7.5}{n\pi}\left[\cos(n\pi) - 1\right] e^{jnt}$

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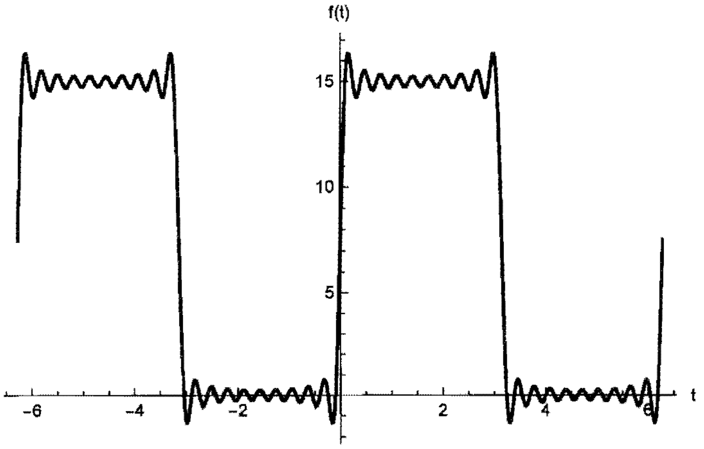
```
In[21]:= ω0 := 1
T := 2 * Pi / ω0
f[nmax_, t_] :=
 7.5 + 7.5 / Pi * Sum[ⅈ / n * (Cos[n * Pi] - 1) * Exp[ⅈ * n * t], {n, -nmax, -1, 1}] +
  7.5 / Pi * Sum[ⅈ / n * (Cos[n * Pi] - 1) * Exp[ⅈ * n * t], {n, 1, nmax, 1}]
Plot[f[20, t], {t, -T, T}, AxesLabel → {"t", "f(t)"}]
```

Out[24]=



$\underline{N_{max} = 20}$ ($\Rightarrow$ 41 terms)

$$f(t) = 7.5 + \sum_{n=-n_{max}}^{-1} \frac{j\,7.5}{n\pi}\left[\cos(n\pi) - 1\right] e^{jnt}$$
$$+ \sum_{n=1}^{n_{max}} \frac{j\,7.5}{n\pi}\left[\cos(n\pi) - 1\right] e^{jnt}$$

Solution 17.16

The waveform in Fig. 17.55(a) has the following Fourier series:

$$v(t) = \frac{1}{2} - \frac{4}{\pi^2}\left(\cos \pi t + \frac{1}{9}\cos 3\pi t + \frac{1}{25}\cos 5\pi t + \cdots\right)V$$

Obtain the Fourier series of $v_2(t)$ in Fig. 17.55(b).

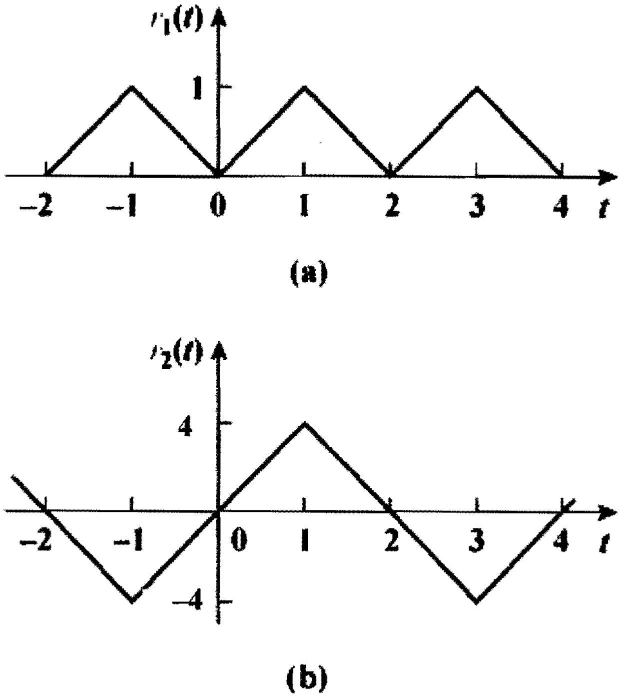
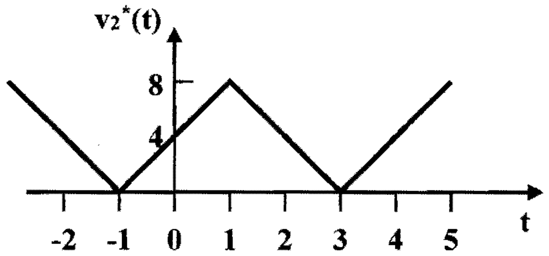


Figure 17.55
For Prob. 17.16.

Solution

If $v_2(t)$ is shifted by 4 along the vertical axis, we obtain $v_2^*(t)$ shown below, i.e. $v_2^*(t) = v_2(t) + 4$.



To replicate v_2^* from v_1:

Comparing $v_2^*(t)$ with $v_1(t)$ shows that

scale by 8 / advance in time / halve the frequency

$$v_2^*(t) = 8v_1((t + t_o)/2)$$

where $(t + t_o)/2 = 0$ at $t = -1$ or $t_o = 1$

Hence $\quad v_2^*(t) = 8v_1((t + 1)/2)$

But $\quad v_2^*(t) = v_2(t) + 4$

$$v_2(t) + 4 = 8v_1((t+1)/2)$$

sub $v_1(t)$; replace $t \to \frac{t+1}{2}$

$$v_2(t) = -4 + 8v_1((t+1)/2)$$

$$\therefore v_2(t) = -4 + 4 - \frac{32}{\pi^2}\left[\cos\pi\left(\frac{t+1}{2}\right) + \frac{1}{9}\cos 3\pi\left(\frac{t+1}{2}\right) + \frac{1}{25}\cos 5\pi\left(\frac{t+1}{2}\right) + \cdots\right]$$

$$v_2(t) = -\frac{32}{\pi^2}\left[\cos\left(\frac{\pi t}{2} + \frac{\pi}{2}\right) + \frac{1}{9}\cos\left(\frac{3\pi t}{2} + \frac{3\pi}{2}\right) + \frac{1}{25}\cos\left(\frac{5\pi t}{2} + \frac{5\pi}{2}\right) + \cdots\right]$$

$$v_2(t) = +\frac{32}{\pi^2}\left[\sin\left(\frac{\pi t}{2}\right) - \frac{1}{9}\sin\left(\frac{3\pi t}{2}\right) + \frac{1}{25}\sin\left(\frac{5\pi t}{2}\right) - \cdots\right]$$

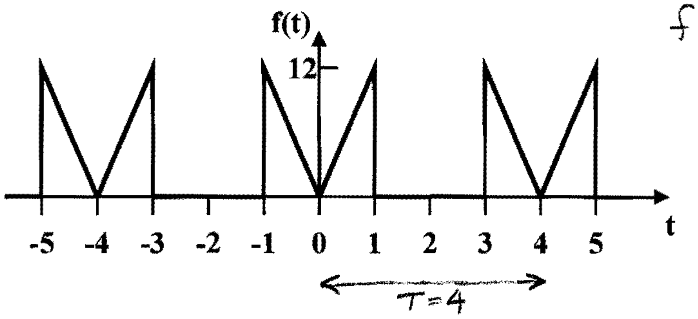
Check. At $t=0$, $v_1 = 0$; $v_2 = -4 + 8 \cdot v_1(\frac{1}{2}) = -4 + 8 \cdot \frac{1}{2} = 0$ ✓

At $t=1$, $v_1 = 1$; $v_2 = -4 + 8 \cdot v_1(1) = -4 + 8 \cdot 1 = 4$ ✓

Solution 17.22

$f(t) = a_o + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t)\right]$

Calculate the Fourier coefficients for the function in Fig. 17.60.



$f(t)$

Figure 17.60
For Prob. 17.60

Solution

This is an even function, therefore $\underline{b_n = 0}$. In addition, T=4 and $\omega_o = \pi/2$.

using (17.18):
$$\underline{a_o} = \frac{2}{T}\int_0^{T/2} f_e(t)dt = \frac{2}{4}\int_0^1 12tdt = 3t^2\Big|_0^1 = 3$$

using (17.18)
$$a_n = \frac{4}{T}\int_0^{T/2} f_e(t)\cos(\omega_o nt)dt = \frac{4}{4}\int_0^1 12t\cos(n\pi t/2)dt$$

$\int x\cos(ax)dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$

$$= 12\left[\frac{4}{n^2\pi^2}\cos(n\pi t/2) + \frac{2t}{n\pi}\sin(n\pi t/2)\right]\Bigg|_0^1$$

$$a_n = \frac{48}{n^2\pi^2}\left[\cos(n\pi/2) - 1\right] + \frac{24}{n\pi}\sin(n\pi/2)$$

Problem 17.23 w/ $f(0) = 1$ and $t_2 = T = 2$

Find the Fourier series of the function shown in Fig. 17.61. $f(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right]$

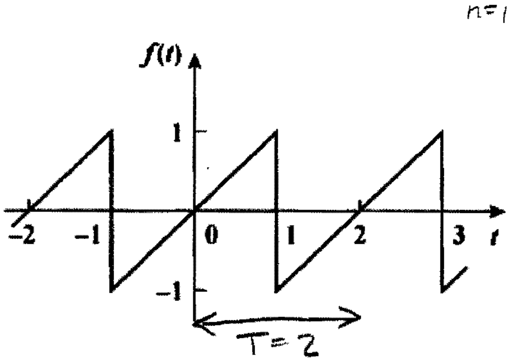


Figure 17.61

Solution

f(t) is an odd function. From (17.28): $\underline{\underline{a_o = 0}}$, $\underline{\underline{a_n = 0}}$

$f(t) = t, \quad -1 < t < 1$

$$a_o = 0 = a_n, \quad T = 2, \quad \omega_o = 2\pi/T = \pi$$

From (17.28): $\underline{\underline{b_n}} = \frac{4}{T}\int_0^{T/2} f(t)\sin(n\omega_o t)dt = \frac{4}{2}\int_0^1 t\sin(n\pi t)dt$ $\int x\sin(ax)dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$

$$= \frac{2}{n^2\pi^2}\left[\underbrace{\sin(n\pi t)}_{\to 0} - n\pi t\cos(n\pi t)\right]_0^1 = -\frac{2n\pi}{(n\pi)^2}\cos(n\pi t)\Big|_0^1 = -\frac{2}{n\pi}\left[\underbrace{\cos(n\pi)}_{(-1)^n} - 0\right]$$

$$= -[2/(n\pi)]\cos(n\pi) = 2(-1)^{n+1}/(n\pi)$$

$$f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi t)$$

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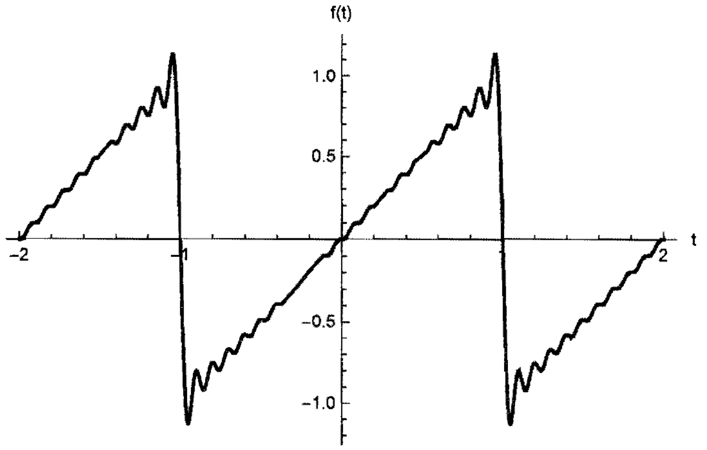
In[41]:=
```
ω0 := Pi
T := 2 * Pi / ω0
f[nmax_, t_] := 2 / Pi * Sum[(-1)^(n+1) / n * Sin[n * Pi * t], {n, 1, nmax}]
Plot[f[20, t], {t, -T, T}, AxesLabel → {"t", "f(t)"}]
```

Out[44]=

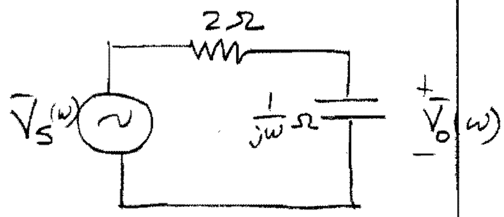


$N_{max} = 20$

Practice Problem 17.6

From Practice Problem 17.2, $v_S(t) = 4.5 - \frac{9}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin(\underbrace{2n\pi}_{\omega_0 = 2\pi} t)$ V

The equivalent phasor domain circuit is



$$\bar{V}_O(\omega) = \frac{\frac{1}{j\omega}}{\frac{1}{j\omega}+2}\cdot \bar{V}_S(\omega) = \frac{1}{1+j2\omega}\bar{V}_S(\omega)$$

- **DC**: Capacitor acts as an open circuit $\Rightarrow$ $\bar{V}_O\big|_{dc} = 4.5$ V

- **AC**: The n^{th} harmonic of $v_S(t)$ is

$$v_{S,n}(t) = -\frac{9}{n\pi}\sin(2n\pi t)$$

choose sin( ) as our phasor form.

phasor: $\bar{V}_{S,n} = -\frac{9}{n\pi}$

$$\therefore \bar{V}_{O,n} = \frac{1}{1+j2\cdot 2n\pi}\left(-\frac{9}{n\pi}\right) = \frac{9}{n\pi\sqrt{1^2+(4n\pi)^2}} \angle -\left[-\tan\left(\frac{4n\pi}{1}\right)\right]$$

$$= \frac{9}{n\pi\sqrt{1+16n^2\pi^2}} \angle \tan^{-1}(4n\pi)$$

$$v_{O,n}(t) = \mathrm{Im}\left\{\bar{V}_{O,n} e^{j\omega t}\right\} = \frac{9}{n\pi\sqrt{1+16n^2\pi^2}}\sin\left[2n\pi t + \tan^{-1}(4n\pi)\right]$$

$$\therefore v_O(t) = 4.5 + \frac{9}{\pi}\sum_{n=1}^{\infty} \frac{\sin\left[2n\pi t + \tan^{-1}(4n\pi)\right]}{n\sqrt{1+16n^2\pi^2}}$$

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Solution 17.32

Find $i(t)$ in the circuit of Fig. 17.68 given that

$$i_s(t) = \left[3.5 + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos(3nt)\right] A$$

$\omega_n = n\omega_0 = n3 \ \frac{rad}{s}$

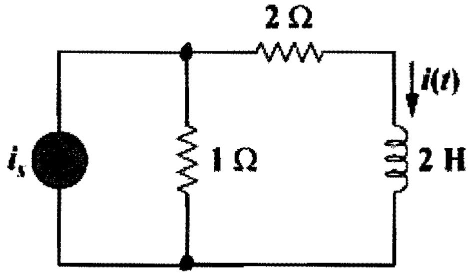
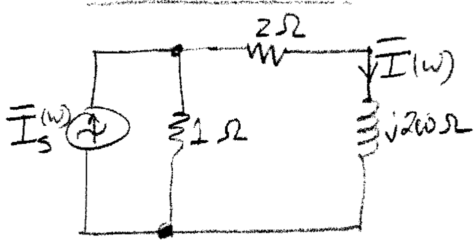


Figure 17.68
For Prob. 17.32.

Phasor domain



By current division:

$$I(\omega) = \frac{1}{1+2+j2\omega} \cdot I_s(\omega) = \frac{1}{3+j2\omega} I_s(\omega)$$

Solution

When $i_s = 3.5$ A, (DC component) – inductor acts as a short circuit.

By current division; $I_{dc} = 3.5/(1+2) = 3.5/3$

For $n \geq 1$, $\omega_n = 3n$, $\bar{I}_s = 4/n^2\angle 0°$ A ← From given $i_s(t)$

From phasor domain ckt using current division: $\bar{I} = [1/(1+2+j2\omega_n)]I_s = \bar{I}_s/(3+j6n)$ ← w/ $\omega = n\omega_0 = n3$

$$\underset{\text{Sub } I_s}{=} \underbrace{\frac{\frac{4}{n^2}\angle 0°}{3\sqrt{1+4n^2}\angle \tan^{-1}(6n/3)}}_{\text{magnitude/angle form}} = \frac{4}{3n^2\sqrt{1+4n^2}}\angle -\tan^{-1}(2n)$$

Thus,

$$i(t) = \left[1.1667 + \sum_{n=1}^{\infty}\frac{4}{3n^2\sqrt{1+4n^2}}\cos(3n - \tan^{-1}(2n))\right] \mathbf{A}$$

$$I_n(\omega) = \frac{I_s}{3+j6n} = \frac{4/n^2}{\sqrt{3^2+(6n)^2}\angle \tan^{-1}(\frac{6n}{3})} = \frac{4/n^2}{\sqrt{9+36n^2}\angle \tan^{-1}(2n)} = \frac{4\angle -\tan^{-1}(2n)}{3n^2\sqrt{1+4n^2}}$$

$$i_n(t) = Re\{I_n(\omega)e^{j\omega t}\} = \frac{4}{3n^2\sqrt{1+4n^2}}\cos[\underset{\overset{\shortparallel}{3n}}{\omega_n} - \tan^{-1}(2n)]$$

Solution 17.33

In the circuit shown in Fig. 17.69, the Fourier series expansion of $v_s(t)$ is

$$v_s(t) = \left[10 + \frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(\underbrace{n\pi t}_{\omega_n})\right]\text{V}$$

$= n\omega_0 = n\pi \ \frac{rad}{s}$

Find $v_o(t)$.

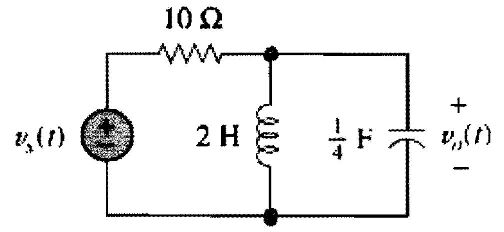
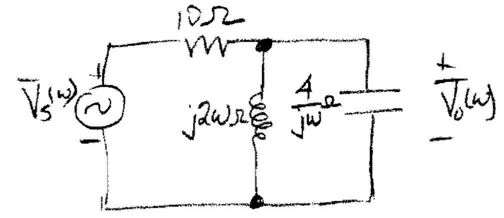


Figure 17.69 For Prob. 17.33.

Phasor domain ckt:



From the Fourier series for $v_s(t)$, we can identify

$\bar{V}_s(\omega) = \frac{5}{n\pi}$ assuming a $\sin(\ )$ phasor reference.

Solution

For the DC case, the inductor acts like a short, $V_o|_{dc} = 0$.
For the AC case, we obtain the following:

KCL: $$\frac{\bar{V}_o - \bar{V}_s}{10} + \frac{\bar{V}_o}{j2\underbrace{n\pi}_{\omega_n}} + \frac{j\overbrace{n\pi}^{\omega_n}\bar{V}_o}{4} = 0$$

or $$\frac{\bar{V}_o}{10} - \frac{j\bar{V}_o}{2n\pi} + \frac{jn\pi\bar{V}_o}{4} = \frac{\bar{V}_s}{10}$$

$$\bar{V}_o - j\frac{10\bar{V}_o}{2n\pi} + \frac{jn\pi 10\bar{V}_o}{4} = \bar{V}_s$$

or $$\left[1 + j\left(2.5n\pi - \frac{5}{n\pi}\right)\right]\bar{V}_o = \bar{V}_s$$

$$\left(1 + j\left(2.5n\pi - \frac{5}{n\pi}\right)\right)\bar{V}_o = \bar{V}_s$$

$$\bar{V}_o = \frac{\bar{V}_s}{1 + j\left(2.5n\pi - \frac{5}{n\pi}\right)}$$

$$\bar{V}_o(\omega) = A_n\angle\Theta_n = \overbrace{\frac{5}{n\pi}}^{\bar{V}_s}\,\frac{1}{1 + j\left(2.5n\pi - \frac{5}{n\pi}\right)} = \frac{5}{n\pi + j(2.5n^2\pi^2 - 5)}$$

$$A_n = \frac{5}{\sqrt{n^2\pi^2 + (2.5n^2\pi^2 - 5)^2}}; \quad \Theta_n = -\tan^{-1}\left(\frac{2.5n^2\pi^2 - 5}{n\pi}\right)$$

$$v_o(t) = \sum_{n=1}^{\infty} A_n \sin(n\pi t + \Theta_n)V$$

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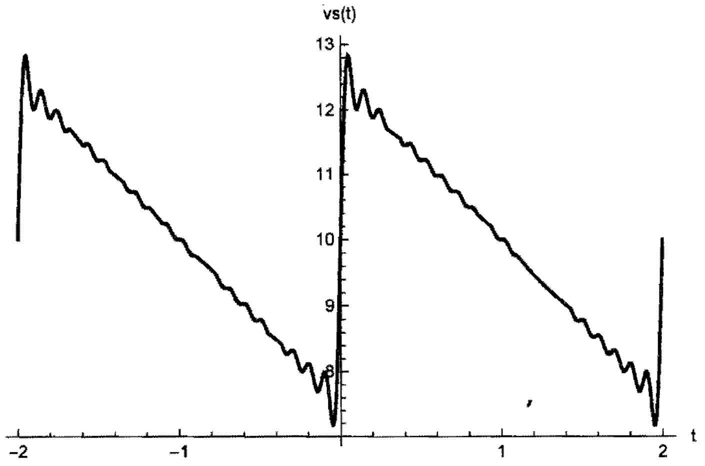
In[53]:=
```
ω0 := Pi
T := 2 * Pi / ω0
vs[nmax_, t_] := 10 + 5 / Pi * Sum[1 / n * Sin[n * Pi * t], {n, 1, nmax}]

A[n_] := 5 / Sqrt[n^2 * Pi^2 + (2.5 * n^2 * Pi^2 - 5)^2]
θ[n_] := -ArcTan[(2.5 * n^2 * Pi^2 - 5) / (n * Pi)]
vo[nmax_, t_] := Sum[A[n] * Sin[n * Pi * t + θ[n]], {n, 1, nmax}]

Plot[vs[20, t], {t, -T, T}, AxesLabel → {"t", "vs(t)"}]
Plot[vo[20, t], {t, -T, T}, AxesLabel → {"t", "vo(t)"}]
```

← ArcTan results are in radians.

Out[59]=



$N_{max} = 20$

Out[60]=