Derive (17.9) showing every step.

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right] \qquad (17.3), (1)$$

multiply (1) by $\sin(m\omega_0 t)$ $m = 1, 2, \cdots$, and integrate over one period of $\omega_0 = \frac{2\pi}{T}$:

$$\int_0^T f(t) \sin(m\omega_0 t)\, dt = \int_0^T a_0 \sin(m\omega_0 t)\, dt + \int_0^T \left\{ \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right] \right\} \sin(m\omega_0 t)\, dt \qquad (2)$$

The first term in the RHS of (2) is zero, since the average value of $\sin(m\omega_0 t)$ over one period T will be zero $\forall$ values of m.

For the second term of the RHS of (2), we can interchange order of summation & multiplication.

$$\int_0^T f(t) \sin(m\omega_0 t)\, dt = \int_0^T \left\{ \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) \sin(m\omega_0 t) + b_n \sin(n\omega_0 t) \sin(m\omega_0 t) \right] \right\} dt \qquad (3)$$

then interchange the order of summation & integration:

$$\int_0^T f(t) \sin(m\omega_0 t)\, dt = \sum_{n=1}^{\infty} \left\{ \int_0^T a_n \cos(n\omega_0 t) \sin(m\omega_0 t)\, dt + \int_0^T b_n \sin(n\omega_0 t) \sin(m\omega_0 t)\, dt \right\} \qquad (4)$$

the coeffs a_n & b_n aren't functions of time, hence can be moved outside the integral operator. Further, we can use the orthogonality of the cosine & sine functions:

$$\int_0^T \cos(n\omega_0 t) \sin(m\omega_0 t)\, dt = 0. \qquad (5)$$

&

$$\int_0^T \sin(n\omega_0 t) \sin(m\omega_0 t)\, dt = \begin{cases} 0 & m \neq n \\ \frac{T}{2} & m = n \end{cases} \qquad (6)$$

using (5) & (6) in (4) gives

$$\int_0^T f(t)\sin(m\omega_0 t)\,dt = \sum_{n=1}^{\infty} b_n \int_0^T \sin(n\omega_0 t)\sin(m\omega_0 t)\,dt$$

$$= b_m \cdot \frac{T}{2}$$

∴, we can solve for b_m:

$$b_m = \frac{2}{T}\int_0^T f(t)\sin(m\omega_0 t)\,dt$$

Since m & n are arbitrary indices and sum over equivalent ranges, we'll interchange m & n yielding:

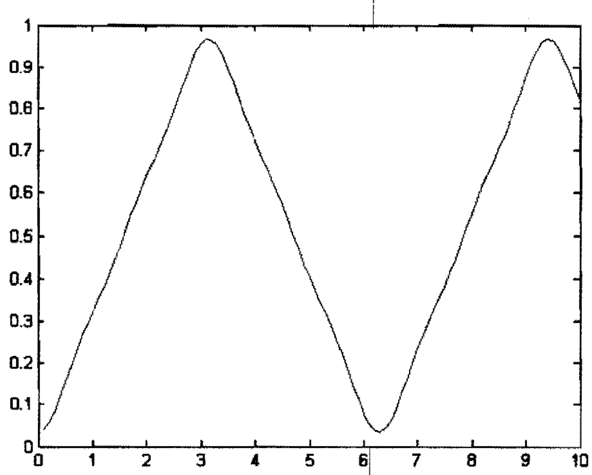
$$\underline{\underline{b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)\,dt}}$$

QED

Solution 17.2

The function f(t) has a DC offset and is even. We use the following MATLAB code to plot f(t). The plot is shown below. If more terms are taken, the curve is clearly indicating a triangular wave shape which is easily represented with just the DC component and three, cosinusoidal terms of the expansion.

```
for n=1:100
   tn(n)=n/10;
   t=n/10;
   y1=cos(t);
   y2=(1/9)*cos(3*t);
   y3=(1/25)*cos(5*t);
   factor=4/(pi*pi);
   y(n)=0.5- factor*(y1+y2+y3);
end
plot(tn,y)
```



Prob. 17.2 (cont)

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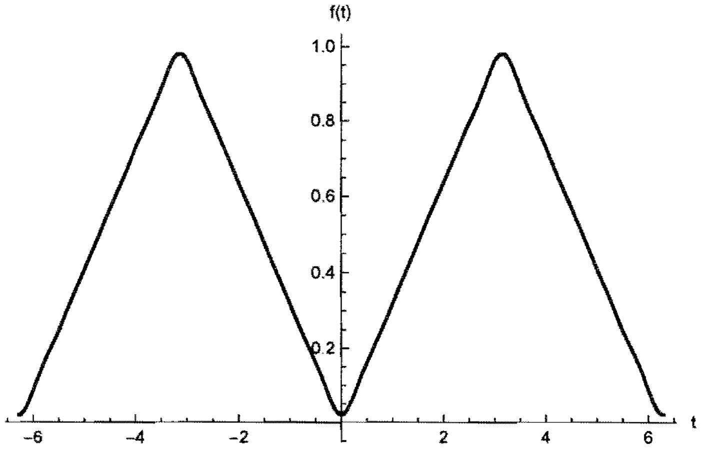
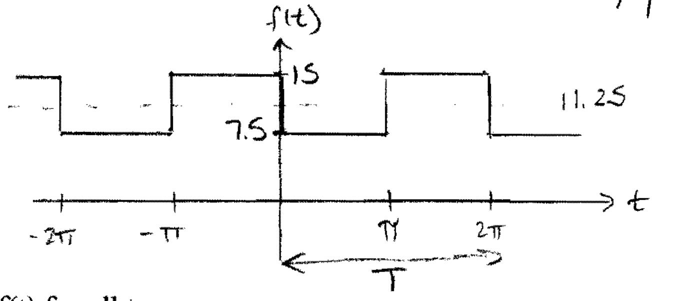
In[15]:=
```
ω0 := 1
T := 2 * Pi / ω0
f[nmax_, t_] := 1 / 2 - 4 / Pi^2 * Sum[1 / n^2 * Cos[n * t], {n, 1, nmax, 2}]
Plot[f[10, t], {t, -T, T}, AxesLabel → {"t", "f(t)"}]
```

Out[18]=



DC + 5 harmonics.



Solution 17.6

Find the trigonometric Fourier series for

$$f(t) = \begin{cases} 7.5, & 0 < t < \pi \\ 15, & \pi < t < 2\pi \end{cases} \quad \text{and } f(t+2\pi)=f(t) \text{ for all t.}$$

$$f(t) = a_o + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega_o t) + b_n\sin(n\omega_o t)\right]$$

Solution

T=2π, $\omega_o=2\pi/T = 1$

(17.6) : $$a_o = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2\pi}\left[\int_0^\pi 7.5dt + \int_\pi^{2\pi} 15dt\right] = \frac{1}{2\pi}(7.5\pi + 15\pi) = 11.25$$

(17.8) : $$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_o t)dt = \frac{2}{2\pi}\left[\int_0^\pi 7.5\cos(nt)\,dt + \int_\pi^{2\pi} 15\cos(nt)\,dt\right] = \frac{7.5}{\pi}\frac{1}{n}\sin(nt)\Big|_0^{\pi} + \frac{15}{\pi}\frac{1}{n}\sin(nt)\Big|_{\pi}^{2\pi} = 0$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_o t)\,dt = \frac{2}{2\pi}\left[\int_0^\pi 7.5\sin(nt)\,dt + \int_\pi^{2\pi} 15\sin(nt)\,dt\right]$$

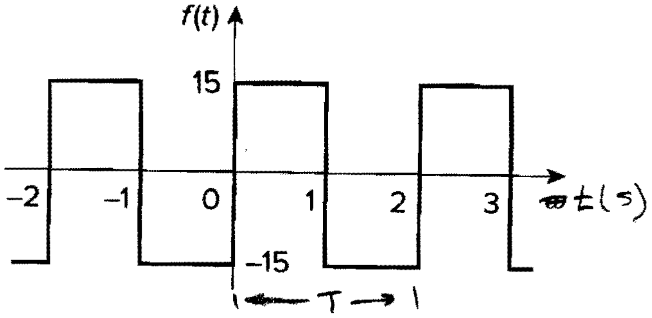
$$= \frac{1}{\pi}\left[-\frac{7.5}{n}\cos(nt)\Big|_0^\pi - \frac{15}{n}\cos(nt)\Big|_\pi^{2\pi}\right]$$ = [7.5/(nπ)][–(cos(nπ)–1) – 2(cos(2nπ)–cos(nπ)] = [7.5/(nπ)][(cos(nπ)+1) – 2] = [7.5/(nπ)][(cos(nπ) – 1].

So for n = even, $b_n = 0$ and for n = odd, $b_n = -15/(n\pi)$.

Thus,

$$f(t) = 11.25 - \sum_{n=\cancel{odd}\ 1}^{\infty}\frac{15}{n\pi}\sin(nt)$$
(n=odd)

$$\frac{7.5}{n\pi}\left\{-\left[\cos(n\pi)-1\right] - 2\left[\cancel{\cos(2n\pi)}^{\,1} - \cos(n\pi)\right]\right\}$$
$$= \frac{7.5}{n\pi}\left\{-\cos(n\pi) + 1 - 2 + 2\cos(n\pi)\right\}$$
$$= \frac{7.5}{n\pi}\left\{\underbrace{\cos(n\pi)}_{=(-1)^n} - 1\right\}$$



P.P.17.1 $T = 2,\ \omega_o = 2\pi/T = \pi$

$$f(t) = \begin{cases} 15, & 0 < t < 1 \\ -15, & 1 < t < 2 \end{cases}$$

$$\underline{a_o} = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{2}\left[\int_0^1 (15)dt + \int_1^2 (-15)dt\right] = 7.5(1-1) = \underline{0}$$

$$\underline{a_n} = \frac{2}{T}\int_0^T f(t)\cos n\omega_o t\,dt = \frac{2}{2}\left[\int_0^1 15\cos n\pi t\,dt + \int_1^2 (-15)\cos n\pi t\,dt\right]$$

$$= \frac{15}{n\pi}[\sin n\pi t]_0^1 - \frac{15}{n\pi}[\sin n\pi t]_1^2 = \underline{0}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_o t\,dt = \frac{2}{2}\left[\int_0^1 15\sin n\pi t\,dt + \int_1^2 (-15)\sin n\pi t\,dt\right]$$

$$= \frac{-15}{n\pi}[\cos n\pi t]_0^1 + \frac{15}{n\pi}[\cos n\pi t]_1^2 = \frac{30}{n\pi}[1 - \cos n\pi]$$

$$b_n = \begin{cases} 60/(n\pi), & \text{for } n = \text{odd} \\ 0, & \text{for } n = \text{even} \end{cases}$$

$$f(t) = a_o + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega_o t) + b_n\sin(n\omega_o t)\right]$$

$$f(t) = \frac{60}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}\sin n\pi t, \qquad \underline{\mathbf{n = 2k-1}}$$

$$= \frac{60}{\pi}\sum_{\substack{n=1 \\ (n\ \text{odd})}}^{\infty}\frac{1}{n}\sin(n\pi t)$$

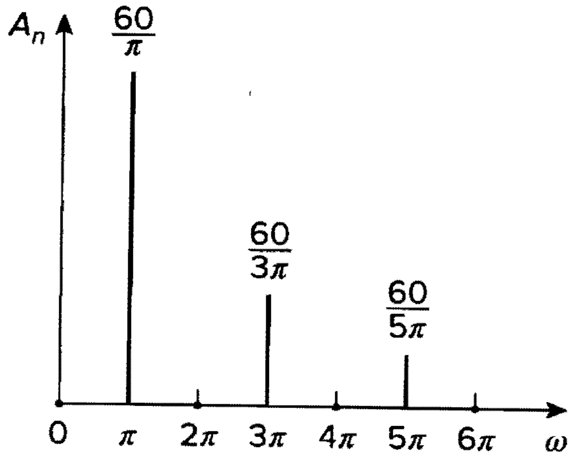
$$= -\frac{15}{n\pi}\left[\cos(n\pi) - 1\right] + \frac{15}{n\pi}\left[\underbrace{\cos(2n\pi)}_{1} - \cos(n\pi)\right]$$

$$= \frac{15}{n\pi}\left[-\cos(n\pi) + 1 + 1 - \cos(n\pi)\right] = \frac{15}{n\pi}\left[2 - 2\cos(n\pi)\right]$$

$$= \frac{30}{n\pi}\left[1 - \underbrace{\cos(n\pi)}_{=(-1)^n}\right]$$

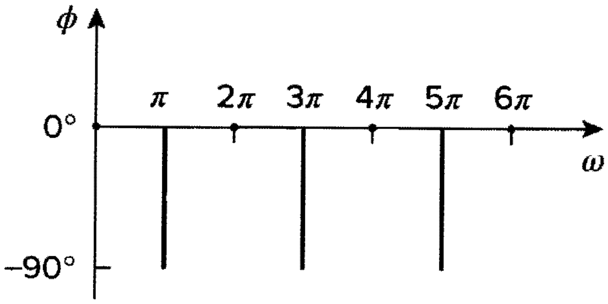
P.P. 17.1 (cont.)

$$A_n = \sqrt{a_n^2 + b_n^2} = |b_n| = \frac{60}{n\pi}, \quad n \text{ odd}$$

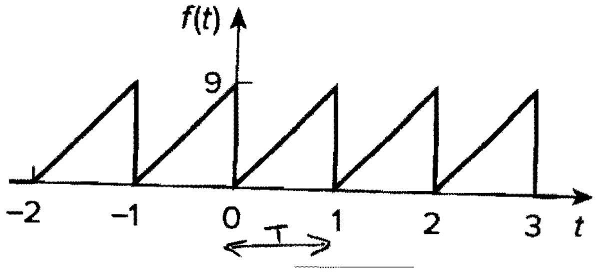


(a)

$$\phi_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right) = -\tan^{-1}(\infty) = -90°$$



(b)



P.P.17.2 $T = 1$, $\omega_o = 2\pi/T = 2\pi$, $f(t) = 6t$, $0 < t < 1$.

(17.6) : $$\underline{\underline{a_o}} = \frac{1}{T}\int_0^T f(t)dt = \left[\int_0^1 (9t)dt\right] = \frac{9t^2}{2}\Big|_0^1 = \underline{\underline{4.5}}$$

$\int x\cos(ax)dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$

(17.8) : $$\underline{\underline{a_n}} = \frac{2}{T}\int_0^T f(t)\cos n\omega_o t\,dt = \frac{2}{1}\left[\int_0^1 9t\cos n2\pi t\,dt\right]$$

$$= 2\left[\frac{9}{(2n\pi)^2}[\cos 2n\pi t] + \frac{9t}{2n\pi}[\sin 2n\pi t]\right]_0^1 = \frac{18}{(2n\pi)^2}\left[\cos(2n\pi) - 1\right] + \frac{9}{2n\pi}\cancel{\sin(2n\pi)} - 0$$

$$= \frac{18}{4n^2\pi^2}[[\cos 2n\pi 1] - 1] = \underline{\underline{0}}$$

$\int x\sin(ax)dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$

(17.9) : $$\underline{\underline{b_n}} = \frac{2}{T}\int_0^T f(t)\sin n\omega_o dt = \frac{2}{1}\left[\int_0^1 9t\sin 2n\pi t\,dt\right]$$

$$= 2\left[\frac{9}{4n^2\pi^2}[\sin 2n\pi t] - \frac{9t}{2n\pi}[\cos 2n\pi t]\right]_0^1 = \frac{-18}{2n\pi}[\cos 2n\pi] = \underline{\underline{-9/(n\pi)}}$$

$$f(t) = a_o + \sum_{n=1}^{\infty}[a_n\cos(n\omega_o t) + b_n\sin(n\omega_o t)]$$

$$\boxed{f(t) = 4.5 - \frac{9}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(2n\pi t)}$$

$$= \frac{18}{2n\pi}\left\{\frac{1}{2n\pi}\left[\cancel{\sin(2n\pi)} - 0\right] - \left[1\cdot\cancel{\cos(2n\pi)} - 0\right]\right\}$$

$$= -\frac{18}{2n\pi} = -\frac{9}{n\pi}$$

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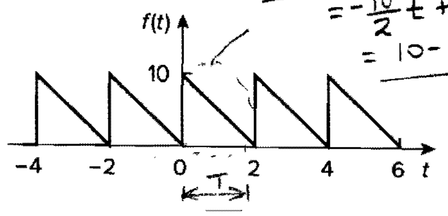
1/1

$f(t) = mt + b$
$= -\frac{10}{2}t + 10$
$= 10 - 5t$



Solution 17.4

$f(t) = 10 - 5t, \ 0 < t < 2, \ T = 2, \ \omega_o = 2\pi/T = \pi$

(17.6): $$\underline{\underline{a_o}} = (1/T)\int_0^T f(t)dt = (1/2)\int_0^2 (10 - 5t)dt = 0.5[10t - (5t^2/2)]\Big|_0^2 = \underline{\underline{5}}$$

(17.8): $$\underline{\underline{a_n}} = (2/T)\int_0^T f(t)\cos(n\omega_o t)dt = (2/2)\int_0^2 (10 - 5t)\cos(n\pi t)dt$$

$$= \underbrace{\int_0^2 (10)\cos(n\pi t)dt}_{=0} - \int_0^2 (5t)\cos(n\pi t)dt$$

$\int x\cos(ax)dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$

$\left(= -\frac{5}{n^2\pi^2}\left[\cos(2n\pi) - 1\right] - \frac{5\cdot 2}{n\pi}\sin(2n\pi)\right)$

$$= \frac{-5}{n^2\pi^2}\cos n\pi t\Big|_0^2 - \frac{5t}{n\pi}\sin n\pi t\Big|_0^2 = [-5/(n^2\pi^2)](\cos 2n\pi - 1) = \underline{\underline{0}}$$

(17.9): $$\underline{\underline{b_n}} = (2/2)\int_0^2 (10 - 5t)\sin(n\pi t)dt$$

$$= \underbrace{\int_0^2 (10)\sin(n\pi t)dt}_{=0} - \int_0^2 (5t)\sin(n\pi t)dt$$

$\int x\sin(ax)dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$

$= -\frac{5}{n^2\pi^2}\left[\sin(2n\pi) - 0\right] + \frac{5\cdot 2}{n\pi}\cos(2n\pi) - 0$

$$= \frac{-5}{n^2\pi^2}\sin n\pi t\Big|_0^2 + \frac{5t}{n\pi}\cos n\pi t\Big|_0^2 = 0 + [10/(n\pi)](\cos 2n\pi) = \underline{\mathbf{10/(n\pi)}}$$

Hence

$$f(t) = a_o + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega_o t) + b_n\sin(n\omega_o t)\right]$$

$$\underline{\underline{f(t) = 5 + \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi t).}}$$