

Solution 16.1

The current in an *RLC* circuit is described by

$$\frac{d^2 i}{dt^2} + 10 \frac{di}{dt} + 25i = 0$$

If $i(0) = 7$ A and $di(0)/dt = 0$, find $i(t)$ for $t > 0$.

Solution

Step 1. Transform the equation into the *s*-domain and solve for $I(s)$.

$$s^2 I(s) - (di(0^-)/dt) - si(0^-) + 10sI(s) - 10i(0^-) + 25I(s) = 0$$

$$(s^2 + 10s + 25)I(s) + [-(di(0^-)/dt) - si(0^-) - 10i(0^-)] = 0$$

$$(s^2 + 10s + 25)I(s) + [-7s - 70] = 0 \text{ or } (s^2 + 10s + 25)I(s) = 7(s + 10) \text{ or}$$

$$I(s) = 7(s + 10)/(s^2 + 10s + 25)$$

Step 2. Perform a partial fraction expansion and then solve for $i(t)$ in the time domain.

$$s^2 + 10s + 25 = 0, \text{ thus } s_{1,2} = \frac{-10 \pm \sqrt{10^2 - 100}}{2} = -5, \text{ repeated roots.}$$

$$I(s) = 7(s + 10)/(s + 5)^2 = A/(s + 5) + B/(s + 5)^2 = (As + A + B)/(s + 5)^2 \text{ or } B = 7(-5 + 10) = 35$$

Multiply out:
Equal powers
of *s*:

$$7s + 70 = As + 5A + B$$

$$A = 7 \text{ and } 5A + B = 70 \text{ or } B = 70 - 35 = 35 \text{ or}$$

$$I(s) = 7/(s + 5) + 35/(s + 5)^2 \text{ or}$$

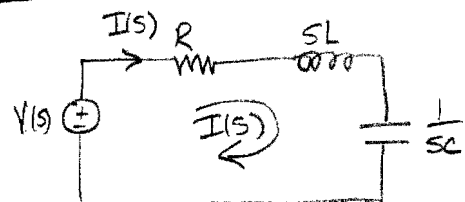
$$A = 7, B = 35$$

$$\text{using Table 15.2: } \underline{i(t) = [(7 + 35t)e^{-5t}]u(t) \text{ A}}$$

Table 15.1:

$$\frac{d^2 f}{dt^2} \longleftrightarrow s^2 F(s) - sf(0^-) - f'(0^-)$$

$$\frac{df}{dt} \longleftrightarrow sF(s) - f(0^-)$$

**Solution 16.4**

If $R = 20 \Omega$, $L = 0.6 \text{ H}$, what value of C will make an RLC series circuit:

- (a) overdamped,
- (b) critically damped,
- (c) underdamped?

Solution

Step 1. Since we are working with a series RLC circuit, we can express our values in terms of $I(s)$ and the s equation that multiplies it in the s -domain. From here we can easily find the values that produce over damped, critically damped, and underdamped conditions.

Equating the mesh equation we get, $RI(s) + LsI(s) + (1/C)I(s)/s - V(s) = 0$ or

$$(0.6s + 20 + 1/(Cs))I(s) = V(s) \text{ or } [s^2 + (20/0.6) + 1/(0.6Cs)]I(s) = V(s)/0.6 \text{ or}$$

$$[s^2 + (20/0.6)s + 1/(0.6C)]I(s) = sV(s)/0.6$$

By KVL: $RI(s) + SLI(s) + \frac{I(s)}{sC} = V(s)$
 $\Rightarrow s^2 L I(s) + sRI(s) + \frac{I(s)}{C} = sV(s)$
 $s^2 I(s) + s \frac{R}{L} I(s) + \frac{I(s)}{LC} = \frac{s}{L} V(s)$
 $\Rightarrow I(s) = \frac{\frac{s}{L} V(s)}{s^2 + s \frac{R}{L} + \frac{1}{LC}} \quad a_1 = 1, b_1 = \frac{R}{L}, c_1 = \frac{1}{LC}$

$$s_{1,2} = \frac{-b_1 \pm \sqrt{b_1^2 - 4a_1c_1}}{2a_1} = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2}$$

$$\text{The roots for the denominator are } s_{1,2} = \frac{-(20/0.6) \pm \sqrt{(400/0.36) - 4/(0.6C)}}{2}$$

Step 2. To find the values of our roots that produces overdamped, critically damped, and underdamped conditions, we note that when s_1 and s_2 values that produces these values,

See Section 8.4:

- overdamped is when s_1 and s_2 are real with no complex values
- critically damped is when $s_1 = s_2$
- underdamped is when both s_1 and s_2 have complex roots and $s_1 = s_2^*$

Now all we need to do is to solve for these conditions.

- (a) Overdamped is when $[4/(0.6C)]$ is less than $400/0.36$ or $(\frac{R}{L})^2 > \frac{4}{LC} \Rightarrow C > \frac{4}{L \cdot (\frac{R}{L})^2} = \frac{4L}{R^2}$
 $C > 4 \times 0.36 / (400 \times 0.6) = 6 \times 10^{-3}$, or $C > \underline{\underline{6 \text{ mF}}}$
- (b) Critically damped is when $[4/(0.6C)]$ is equal to $400/0.36$ or $(\frac{R}{L})^2 = \frac{4}{LC} \Rightarrow C = \frac{4}{L \cdot (\frac{R}{L})^2} = \frac{4L}{R^2}$
 $C = 4 \times 0.36 / (400 \times 0.6) = 6 \times 10^{-3} = \underline{\underline{6 \text{ mF}}}$
- (c) Underdamped is when $[4/(0.6C)]$ is greater than $400/0.36$ or $(\frac{R}{L})^2 < \frac{4}{LC} \Rightarrow C < \frac{4L}{R^2}$
 $C < 4 \times 0.36 / (400 \times 0.6) = 6 \times 10^{-3}$ or $C < \underline{\underline{6 \text{ mF}}}$

Solution 16.11

The step response of a parallel RLC circuit is

$$v = 10 + 20e^{-300t} (\cos 400t - 2 \sin 400t) \text{ V}, t \geq 0$$

when the inductor is 50 mH. Find R and C.

Solution

Step 1. There are different ways to approach this problem so, we will convert everything into the s-domain and then solve for the unknowns. We should also note that the steady-state voltage is 10 volts, then the circuit is a step input voltage across a parallel combination of a capacitor and an inductor all in series with an output resistor.

(KCL)

The nodal equation for this circuit is given by,

$$\underbrace{I_1}_{(V-10/s)/R} + \underbrace{I_2}_{(V-0)/(0.05s)} + \underbrace{I_3}_{(V-0)/(1/sC)} = 0 \text{ or}$$

$$[(1/R) + (1/(0.05s)) + sC]V = 10/(Rs) = [(20R + RCs^2 + s)/(Rs)]V \text{ or}$$

$$V = [10/(Rs)] [Rs / (RCs^2 + s + 20R)] = 10 / [(RCs)(s^2 + (1/(RC))s + (20/C))]$$

Step 2. From the value of v(t), we can determine the value of the roots of the polynomial $(s^2 + (1/(RC))s + (20/C)) = (s+300+j400)(s+300-j400)$ thus,
 $20/C = 300^2 + 400^2 = 90,000 + 160,000 = 250,000$ or

and $1/(RC) = 600$ or

$C = 20/250,000 = 80 \mu\text{F}$

$R = 1/(600 \times 80 \times 10^{-6}) = 20.83 \Omega$

$\bullet \frac{1}{LC} = 250,000 \Rightarrow C = (2.5 \times 10^5 \cdot 50 \times 10^{-3})^{-1}$
 $C = 80 \mu\text{F}$

$\bullet \frac{1}{RC} = 600 \Rightarrow R = (600 \cdot 80 \times 10^{-6})^{-1}$
 $R = 20.83 \Omega$

KCL: $\frac{V - \frac{10}{s}}{R} + \frac{V}{sL} + \frac{V}{1/sC} = 0 \text{ or } V(\frac{1}{R} + \frac{1}{sL} + sC) = \frac{10}{sR}$

$$V = \frac{\frac{10}{sR}}{\frac{1}{R} + \frac{1}{sL} + sC}$$

$$V = \frac{\frac{10}{RC}}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

$$\frac{sLR}{sLR} = \frac{10 \cdot L}{s^2RLC + sL + R} \cdot \frac{\frac{1}{RLC}}{\frac{1}{RLC}} \quad \text{Standard form}$$

From Table 15.2:

$$\mathcal{L}\{e^{-300t} \cos 400t\} = \frac{s+300}{(s+300)^2 + (400)^2}$$

$$(s+300)^2 + 400^2 = s^2 + 600s + 300^2 + 400^2$$

Equate:
 $RC = 600$
 $\frac{1}{LC} = 300^2 + 400^2$

Whites

multiply out: $\frac{7.5s^2}{(s+2)\left[\left(s+\frac{1}{2}\right)^2 + \frac{7}{4}\right]} = \frac{A}{s+2} + \frac{Bs+C}{\left(s+\frac{1}{2}\right)^2 + \frac{7}{4}} =$

$\therefore 7.5s^2 = A\left[\left(s+\frac{1}{2}\right)^2 + \frac{7}{4}\right] + (Bs+C)(s+2) = A(s^2+s+2) + Bs^2 + B2s + Cs + 2C$

Solution 16.17

$s^2: 7.5 = A + B \Rightarrow B = 7.5 - A = 0$

$s^0: 2A + 2C = D = C = -A = -7.5$

If $i_s(t) = 7.5e^{-2t}u(t)$ A in the circuit shown in Fig. 16.40, find the value of $i_o(t)$.

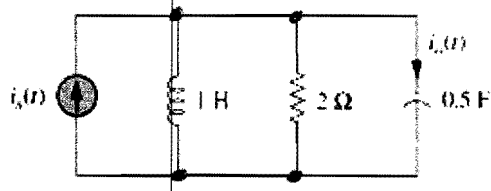
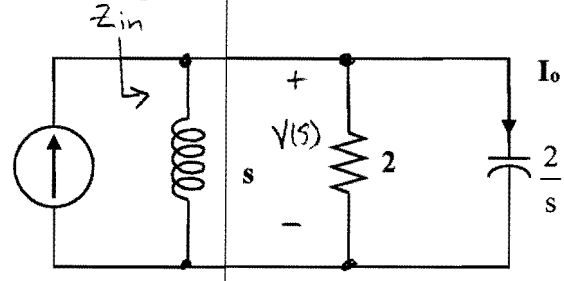


Figure 16.40 For Prob. 16.17.

Solution

We need to determine the initial conditions which in this case are all equal to zero since there are no sources before $t = 0$. Next we convert the circuit into the s -domain. We can write a nodal equation and then calculate $I_o = V/(2/s)$. We then perform a partial fraction expansion and convert back into the time domain.



$V(s) = Z_{in}(s) \cdot I_s(s)$

$V(s) = \frac{7.5}{s+2} \left(\frac{1}{s} + \frac{1}{2} + \frac{s}{2} \right) = \frac{7.5}{s+2} \left(\frac{2s}{s^2+s+2} \right) = \frac{15s}{(s+2)(s+0.5+j1.3229)(s+0.5-j1.3229)}$

complex roots

$I_o = \frac{Vs}{2} = \frac{7.5s^2}{(s+2)(s+0.5+j1.3229)(s+0.5-j1.3229)}$

$= \frac{7.5}{s+2} + \frac{7.5(-0.5-j1.3229)^2}{(1.5-j1.3229)(-j2.646)} + \frac{7.5(-0.5+j1.3229)^2}{(1.5+j1.3229)(+j2.646)}$

$i_o(t) = 7.5(e^{-2t} + 0.3779e^{-90^\circ}e^{-t/2}e^{-j1.3229t} + 0.3779e^{90^\circ}e^{-t/2}e^{j1.3229t})u(t) A$

or $\therefore I(s) = \frac{7.5}{s+2} - \frac{7.5}{\left(s+\frac{1}{2}\right)^2 + \frac{7}{4}}$
 $= \frac{7.5}{s+2} - 7.5 \cdot \frac{2}{7} \cdot \frac{\left(\frac{\sqrt{7}}{2}\right)^2}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2}$
 $= 7.5(e^{-2t} - 0.7559e^{-0.5t} \sin 1.3229t)u(t) A$

Partial Fraction Expansion (PFE):
 $= \frac{A}{s+2} + \frac{Bc + D}{\left(s+\frac{1}{2}\right)^2 + \frac{7}{4}}$
 $\Rightarrow A = \frac{7.5 \cdot (-2)^2}{(-2+\frac{1}{2})^2 + \frac{7}{4}}$

$A = 7.5$
 $\leftarrow B = 0, C = -A$

or $i_o(t) = 7.5 \left(e^{-2t} - \frac{2}{\sqrt{7}} e^{-0.5t} \sin \left(\frac{\sqrt{7}}{2} t \right) \right) u(t) A$

Solution 16.18

Find $v(t)$, $t > 0$ in the circuit of Fig. 16.41. Let $v_s = 12$ V.

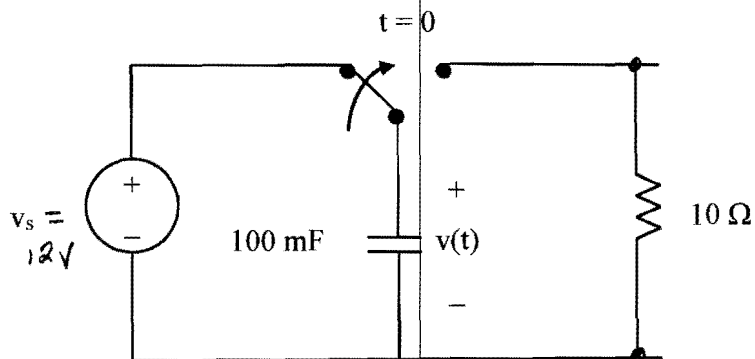
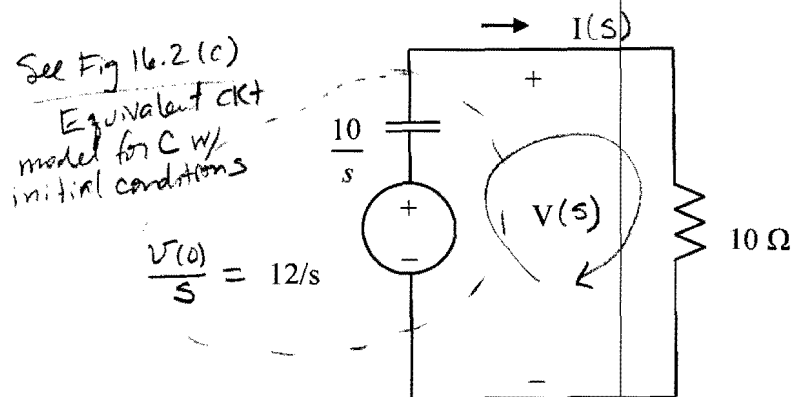


Figure 16.41
For Prob. 16.18.

Solution

For $t < 0$, $v(0) = v_s = 12$ V.

For $t > 0$, the circuit in the s -domain is as shown below.



$$\begin{aligned} \text{KVL: } \frac{12}{s} &= I(s) \cdot \frac{10}{s} + 10 \cdot I(s) \\ \Rightarrow I(s) &= \frac{12/s}{10/s + 10} \cdot \frac{s}{10} \\ &= \frac{1.2}{s+1} \end{aligned}$$

$$\begin{aligned} V(s) &= 10 \cdot I(s) = \frac{12}{s+1} \\ \underline{v(t)} &= \mathcal{L}^{-1}\{V(s)\} = \underline{12e^{-t} u(t) \text{ V}} \end{aligned}$$

$$100 \text{ mF} = 0.1 \text{ F} \quad \longrightarrow \quad \frac{1}{sC} = \frac{10}{s} \quad \text{and}$$

$$I = (12/s) / [10 + (10/s)] = 1.2 / (s+1) \quad \text{and} \quad V = 10I \quad \text{which gives us}$$

$$\underline{v(t) = [12e^{-t}] u(t) \text{ V.}}$$

Solution 16.19

The switch in Fig. 16.42 moves from position A to position B at $t=0$ (please note that the switch must connect to point B before it breaks the connection at A, a make before break switch). Find $v(t)$ for $t > 0$.

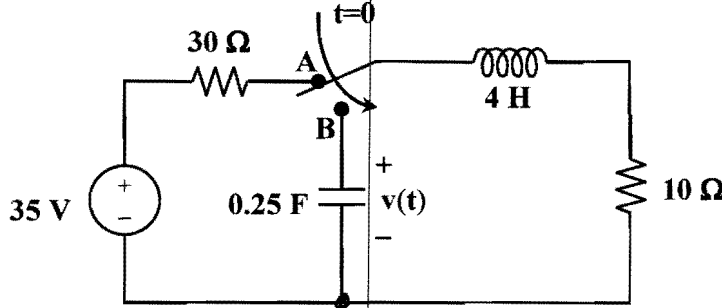


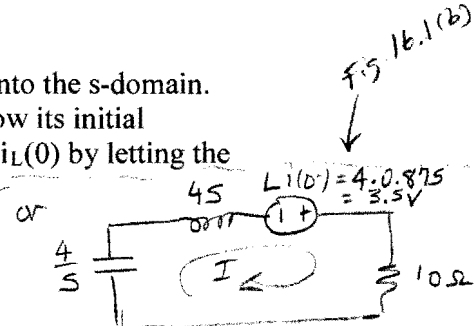
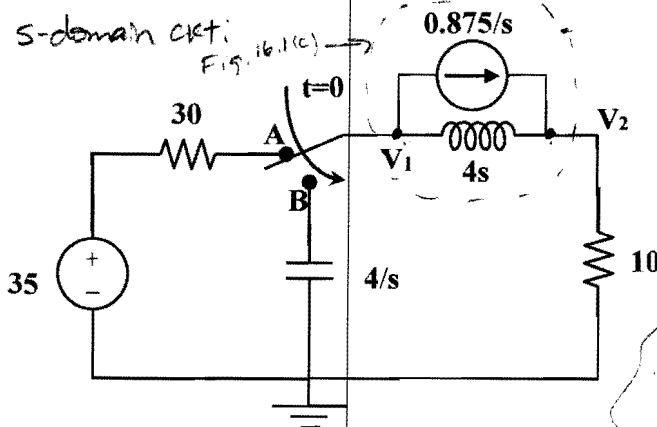
Figure 16.42 For Prob. 16.19.

Solution

Step 1. First find all the initial conditions and then transform into the s-domain.

Since the capacitor is not connected to a circuit, we do not know its initial condition so we can assume it is zero $[v(0) = 0]$. We can find $i_L(0)$ by letting the inductor be a short and $i_L(0) = 35/40 = 0.875$ amp.

Equivalent s-domain ckt. Fig. 16.1(c)



or

$$KVL: \frac{4}{s}I + 4sI + 10I = 3.5$$

$$\text{or } (4 + 4s^2 + 10s)I = 3.5s$$

$$I = \frac{3.5s}{4(s^2 + 2.5s + 1)}$$

$$V = \frac{4}{s}I = \frac{1.4}{(s + \frac{1}{2})(s + 2)}$$

$$V = \frac{3.5}{(s + \frac{1}{2})(s + 2)} = \frac{A}{s + \frac{1}{2}} + \frac{B}{s + 2}$$

$$\Rightarrow A = \frac{3.5}{1.5} = 2.333; B = \frac{3.5}{-1.5} = -2.333$$

$$v(t) = [-2.333e^{-t/2} + 2.333e^{-2t}]u(t) \text{ V}$$

KCL after $t=0$:

$$[(V_1 - 0)/(4/s)] + [(V_1 - V_2)/(4s)] + (0.875/s) = 0 \text{ and}$$

$$[(V_2 - V_1)/(4s)] + (-0.875/s) + [(V_2 - 0)/10] = 0 \text{ where } V = V_1.$$

Next, add these together, $[sV_1/4] + [V_2/10] = 0$ or $V_2 = -2.5sV_1$. Now we can solve for V_1 and V_2 .

Step 2.

$$[(s/4) + (1/(4s)) + (2.5s/(4s))]V_1 = -0.875/s$$

$$= [(s^2 + 2.5s + 1)/(4s)]V_1 \text{ or } V_1 = -0.875(4)/(s^2 + 2.5s + 1) = -3.5/[(s + 0.5)(s + 2)]$$

$$= [-2.3333/(s + 0.5)] + [2.3333/(s + 2)] \text{ or}$$

$$v(t) = [-2.333e^{-t/2} + 2.333e^{-2t}]u(t) \text{ volts.}$$

Solution 16.64

The switch in Fig. 16.87 moves from position 1 to position 2 at $t = 0$. Find $v(t)$, for all $t > 0$.

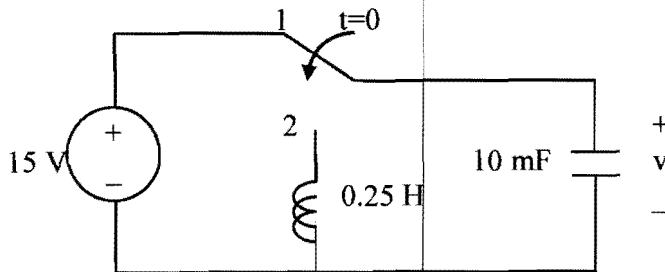
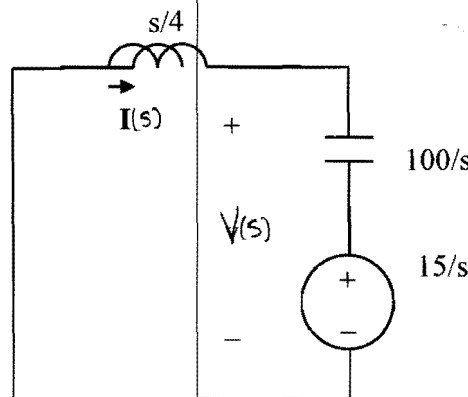


Figure 16.87
For Prob. 16.64.

Solution

When the switch is position 1, $v(0) = 15$ V, and $i_L(0) = 0$. When the switch is in position 2, we have the circuit as shown below, using Fig. 16.2(b)

*equivalent
s-domain*



$$I(s) = -\frac{15/s}{\frac{s}{4} + \frac{100}{s}} \cdot \frac{4s}{4s} = \frac{-60}{s^2 + 400}$$

$$V(s) = -\frac{s}{4} \cdot I(s) = \frac{15s}{s^2 + 400}$$

$$V(s) = \frac{15s}{s^2 + (20)^2}$$

using Table 15.2:

$$\underline{v(t) = \mathcal{L}^{-1}\{V(s)\} = 15 \cos(20t) u(t) \text{ V}}$$

$$I = -(15/s) / [0.25s + (100/s)] = -60 / (s^2 + 400) \text{ and } V = -0.25sI \text{ (or } (100/s)I + 15/s).$$

$$I = 15s / (s^2 + 400) \text{ which leads to,}$$

$$\underline{v(t) = [15 \cos(20t)] u(t) \text{ V}}$$

Solution 16.71

For the ideal transformer circuit in Fig. 16.94, determine $i_o(t)$.

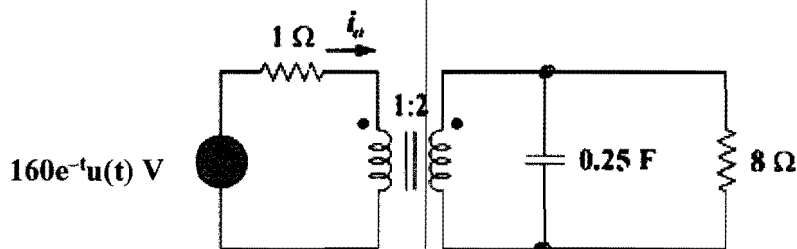
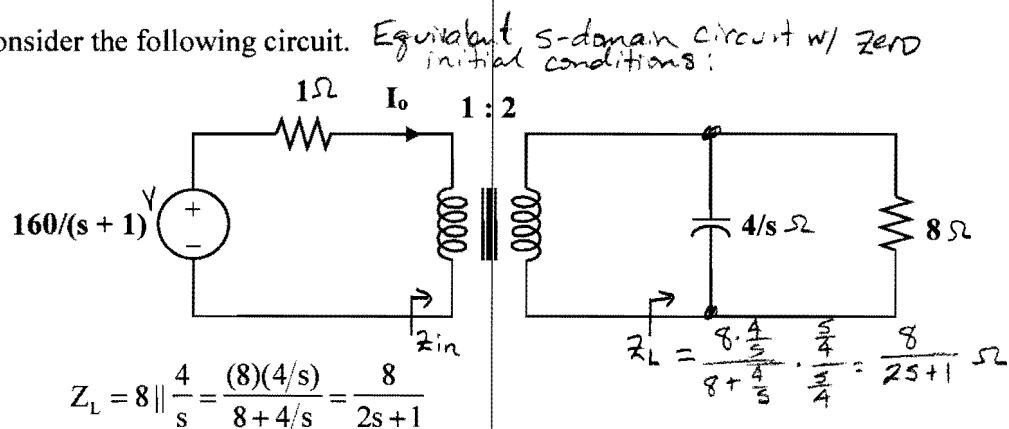


Figure 16.94
For Prob. 16.71.

Solution

Consider the following circuit.



When this is reflected to the primary side,

$$Z_{in} = 1 + \frac{Z_L}{n^2}, \quad n = \frac{2}{1} = 2$$

$$Z_{in} = 1 + \frac{2}{2s+1} = \frac{2s+3}{2s+1}$$

$$I_o = \frac{160}{s+1} \cdot \frac{1}{Z_{in}} = \frac{160}{s+1} \cdot \frac{2s+1}{2s+3} \cdot \frac{1}{\frac{1}{2}}$$

$$I_o = \frac{160s+80}{(s+1)(s+1.5)} = \frac{A}{s+1} + \frac{B}{s+1.5}$$

$$A = -160 \text{ and } B = 320.$$

$$A = \frac{-160+80}{0.5} = \frac{-80}{0.5} = -160$$

$$B = \frac{160(-1.5)+80}{-0.5} = 320$$

$$\therefore I_o(s) = \frac{-160}{s+1} + \frac{320}{s+1.5}$$

Using Table 15.2 :

$$\underline{i_o(t) = 160[2e^{-1.5t} - e^{-t}]u(t) \text{ A.}}$$

Solution 16.83

$$(a) \quad H(s) = \frac{V_o}{V_s} = \frac{R}{R + sL} = \frac{R/L}{s + R/L}$$

$$h(t) = \frac{R}{L} e^{-Rt/L} u(t)$$

$$(b) \quad v_s(t) = u(t) \longrightarrow V_s(s) = 1/s$$

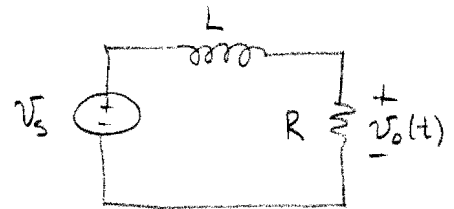
$$V_o = \frac{R/L}{s + R/L} V_s = \frac{R/L}{s(s + R/L)} = \frac{A}{s} + \frac{B}{s + R/L}$$

$$A = 1, \quad B = -1$$

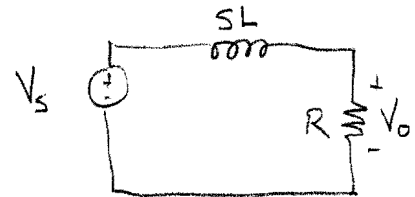
$$V_o = \frac{1}{s} - \frac{1}{s + R/L}$$

$$\underline{v_o(t) = u(t) - e^{-Rt/L} u(t) = (1 - e^{-Rt/L}) u(t)}$$

(9)



Equivalent s domain ckt



By voltage division:

$$V_o = \frac{R}{R + sL} \cdot V_s$$

$$\therefore H(s) \equiv \frac{V_o}{V_s} = \frac{R}{R + sL} \cdot \frac{L}{L}$$

or in standard form

$$H(s) = \frac{R/L}{s + R/L}$$

$$\underline{h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{R}{L} e^{-\frac{R}{L}t} u(t)}$$

(b) Since $V_o(s) = H(s) V(s)$, we first determine $V(s)$

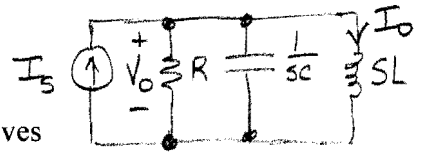
$$\text{With } v_s(t) = u(t) \Rightarrow V_s = \frac{1}{s}$$

$$\therefore V_o(s) = \frac{R/L}{s + R/L} \cdot \frac{1}{s} = \frac{A}{s} + \frac{B}{s + R/L} \quad A = \frac{R/L}{R/L} = 1 \quad B = -\frac{R/L}{R/L} = -1$$

$$\therefore V_o(s) = \frac{1}{s} - \frac{1}{s + R/L}$$

$$\underline{v_o(t) = \mathcal{L}^{-1}\{V_o(s)\} = u(t) - e^{-\frac{R}{L}t} u(t) = (1 - e^{-\frac{R}{L}t}) u(t) \text{ V}}$$

Equivalent s-domain ckt w/ zero initial conditions:

**Solution 16.96**

If V_o is the voltage across R , applying KCL at the non-reference node gives

$$I_s = \frac{V_o}{R} + sC V_o + \frac{V_o}{sL} = \left(\frac{1}{R} + sC + \frac{1}{sL} \right) V_o$$

$$V_o = \frac{I_s}{\frac{1}{R} + sC + \frac{1}{sL}} = \frac{sRL I_s}{sL + R + s^2 RLC}$$

$$I_o = \frac{V_o}{sL} = \frac{R I_s}{s^2 RLC + sL + R}$$

$$H(s) = \frac{I_o}{I_s} = \frac{R}{s^2 RLC + sL + R} = \frac{s/RC}{s^2 + s/RC + 1/LC}$$

The roots

$$s_{1,2} = \frac{-1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

both lie in the left half plane since R , L , and C are positive quantities.

Thus, **the circuit is stable.**

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

✓ if this is real then $< \frac{1}{2RC}$
 $\Rightarrow s_1 < 0$.

✓ if this is imag then
 $\text{Re}\{s_1\} = -\frac{1}{2RC} < 0$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

✓ If this is real then $s_2 < 0$

✓ If this is imag, then

$$\text{Re}\{s_2\} = -\frac{1}{2RC} < 0$$

∴ both roots in the LHP. \Rightarrow stable.

$$I_o = \frac{V_o}{sL} = \frac{R I_s}{sL + R + s^2 RLC}$$

$$H(s) = \frac{I_o}{I_s} = \frac{R}{s^2 RLC + sL + R} \cdot \frac{\frac{1}{RLC}}{\frac{1}{RLC}}$$

$$H(s) = \frac{1/LC}{s^2 + s/RC + 1/LC}$$

$$\text{Roots } s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$s_{1,2} = \frac{-\frac{1}{RC} \pm \sqrt{\left(\frac{1}{RC}\right)^2 - 4/LC}}{2}$$

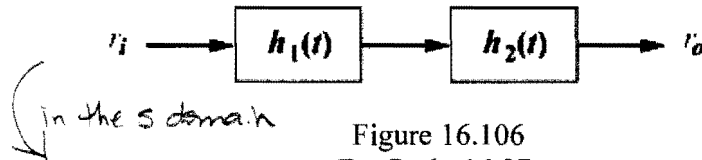
$$= -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

Solution 16.97

A system is formed by cascading two systems as shown in Fig. 16.106. Given that the impulse responses of the systems are,

$$h_1(t) = 21e^{-t} u(t), \quad h_2(t) = e^{-4t} u(t)$$

- (a) Obtain the impulse response of the overall system.
 (b) Check if the overall system is stable.

**Solution**

(a) $H_1(s) = \frac{21}{s+1}, \quad H_2(s) = \frac{1}{s+4}$

$$H(s) = H_1(s)H_2(s) = \frac{21}{(s+1)(s+4)} = [A/(s+1)] + [B/(s+4)] \text{ where}$$

$$A = 21/3 = 7 \text{ and } B = 21/(-3) \text{ therefore,}$$

$$H(s) = \frac{7}{s+1} - \frac{7}{s+4}$$

$$A = \frac{21}{3} = 7$$

$$B = \frac{21}{-3} = -7$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\} \Rightarrow \underline{h(t) = 7[e^{-t} - e^{-4t}]u(t)}$$

- (b) Since the poles of $H(s)$ all lie in the left half s-plane, **the system is stable.**

$$\downarrow$$

$$s_1 = -1$$

$$s_2 = -4$$